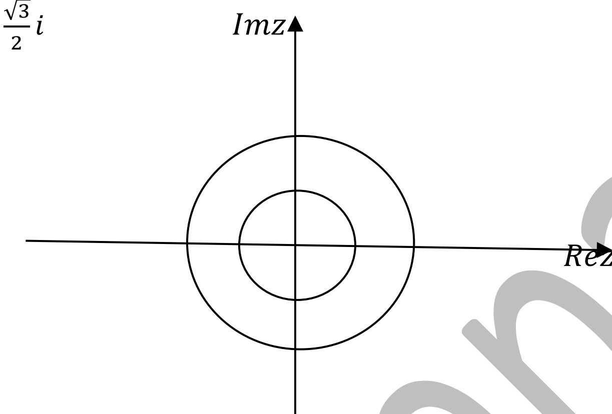
JULY 2009

Exercise 1 (5pts)

1. Let's draw the circles with centre 0 and radius $1\ and\ 2$. indicate the points A; B; C; D, with affixes $\sqrt{3}+i, \sqrt{3}-i,$ and $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$



2) Given the rotation on R with centre 0 and an angle $\frac{\pi}{3}$ and the translation T of a vector with affix I:

a) Let's determine affixes $z_{A'}$ and $z_{B'}$, respective images of points A and B by rotation R.

For $Z_{A'}$ we have $(Z_{A'}-0)=e^{i\frac{\pi}{3}}\left[\left(\sqrt{3}+i\right)-0\right]=e^{i\frac{\pi}{3}}\left(\sqrt{3}+i\right)=2e^{i\frac{\pi}{3}}e^{i\frac{\pi}{3}}=2e^{i\frac{2\pi}{3}}$

For $Z_{B'}$ we have $(Z_{B'}-0)=e^{i\frac{\pi}{3}}\left[\left(\sqrt{3}-i\right)-0\right]=e^{i\frac{\pi}{3}}\left(\sqrt{3}-i\right)=2$. Thus we have

$$Z_{A'}=e^{i\frac{\pi}{3}}\left(\sqrt{3}+i\right)=2e^{i\frac{2\pi}{3}}\ and\ Z_{B'}=e^{i\frac{\pi}{3}}\left(\sqrt{3}-i\right)=2$$

b) Let's find or determine the affix $Z_{D'}$ of point D' image of point D by the translation R.

We have $Z_{D'}=Z_D+a, a=1,\ affix\ of\ vector \Longrightarrow Z_{D'}=\left[-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right]+1=\frac{1}{2}+\frac{\sqrt{3}i}{2}$.

Thus we have

$$Z_{D'}=\frac{1}{2}+\frac{\sqrt{3}i}{2}$$

c)Let's indicate the points $A', B', and\ D'$.

3) let's determine an argument of the complex number $\frac{Z_{A'}-Z_{B'}}{Z_{D'}}=Z_{E'}$

We've $\frac{Z_{A'}-Z_{B'}}{Z_{D'}}=\frac{2e^{i\frac{\pi}{3}}-2}{e^{\frac{\pi i}{6}}}=\frac{2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i-1\right)}{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)}=\frac{2(1+\sqrt{3}i)}{(\sqrt{3}+i)}=\frac{2e^{\frac{\pi i}{3}}}{e^{i\frac{\pi}{6}}}=2e^{i\frac{\pi}{3}}.e^{-i\frac{\pi}{6}}$

$$=2e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)}=2e^{i\frac{\pi}{6}}$$

thus

$$argument\ of\ Z_{E'}=\frac{\pi}{6}$$

Let's prove that the line (OD') is a median of the triangle $OA'B'$

We have $\left|\frac{Z_{D'}-Z_{B'}}{Z_{D'}-Z_{A'}}\right|=\left|\frac{\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-(2)}{\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-2e^{i\frac{2\pi}{3}}}\right|=\left|\frac{1-2}{1-4}\right|=\frac{1}{3}$

$since\ \left|\frac{Z_{D'}-Z_{B'}}{Z_{D'}-Z_{A'}}\right|=\frac{1}{3}$, we conclude that OD' is the median of the triangle $OA'B'$ ■

Exercise 2 (5pts)
Given the differential equation $y' + 5y = 0 \ldots\ldots(E)$.
4)let's prove that the function f is the solution of (E), iff the function $F = f$ is the solution of .
We have $f'' + 5f' = 0 \Longrightarrow f$ is a solution of (E).
If $F = f'$ is a solution of $(E_1) \Longrightarrow F' + 5F = f'' + 5f' = 0$

Hence f *is a solution of* (E) *iif* $F = f'$

5)let's resolve the differential equation (E)
We have $y' + 5y = 0 \Longrightarrow \frac{y'}{y} = -5 \Longrightarrow \int \frac{y'}{y} dy = -\int 5dx$ from where
$\ln|y| = -5x + c \Longrightarrow y = e^{-5x+c} = Ae^{-5x}, (A = e^c) \in \mathbb{R}$.

$y = Ae^{-5x}, A \in \mathbb{R}$

6)Given that $g(x) = a\cos(x) + b\sin(x), a, b, and\ x \in \mathbb{R}$, let's determine $a\ and\ b$ such that g checks the differential equation $y'' + 5y' = 26\cos(x)$
We've $g'(x) = -a\sin(x) + bcos\ and\ g''(x) = -a\cos(x) - b\sin(x)$
Substituting $g'(x) and\ g''(x)\ into\ g(x)$ above, we have
$g''(x) + g'(x) = -[a\cos(x) + b\sin(x)] + 5[-a\sin(x) + b\cos(x)] \equiv 26\cos(x)$ from where we have,
$[-a\cos(x) + 5b\cos(x)] + [-b\sin(x) - 5a\sin(x)\ ] = 26\cos(x)$
$\Longrightarrow (-a + b)\cos(x) + (-a - b)\sin(x) = 26\cos(x)$ Equating coefficients of $\sin(x)$ *and* $\cos(x)$ *on the RHS to that on the LHS* of the equation, we have

$$\begin{cases} -a + 5b = 26 \ldots\ldots(i) \\ -5a - b = 0 \ldots\ldots(ii) \end{cases} \Rightarrow \begin{cases} a = -\frac{b}{5} \ldots\ldots(ii) \\ -\left(-\frac{b}{5}\right) + 5b = 26 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 5 \end{cases}$$

Thus $g(x) = -\cos(x) + 5\sin(x) \equiv a\cos(x) + b\sin(x)$

$a = -1\ and\ b = 5$

7)let's prove that f is a solution of (E') iff $f - g$ is solution of (E).
If f is a solution of $(E') \Longrightarrow f'' + f' = 26\cos(x) \ldots\ldots(iii)$
And if $(f - g)$ is a solution of (E)
$\Longrightarrow (f - g)'' - (f - g)' = 0 \Longrightarrow (f'' - 5f') - (g'' - f') = 0$
From (iii), we've $26\cos(x) - (g'' - 5g') = 0 \Longrightarrow (g'' - 5g') = 26\cos(x) \ldots\ldots(iv)$

$(iii) and\ (iv) \Longrightarrow f$ *is a solution of* E' *iif* $f - g$ *is a solution of* E

8)let's determine all the solutions of E' or the general solution of E'
we have the characteristic equation.
$x^2 + 5x = 0 \Longrightarrow x = 0\ or\ x = -5$
the particular solution is $y_p = -\cos(x) + 5\sin(x)$
The general solution of E' is $f_g = y_p + y_c = -\cos(x) + 5\sin(x) + Ae^{-5x} + B, A, B \in \mathbb{R}$.

Thus $f_g = -\cos(x) + 5\sin(x) + Ae^{-5x} + B\ , A, B \in \mathbb{R}$.

9)let's determine the solution of $E', checking\ f(0) = 0\ and\ f'(0) = 0$.
We've $f(x) = -\cos(x) + 5\sin(x) + Ae^{-5x} + B\ and$
$f'(x) = \sin(x) + 5\cos(x) - 5Ae^{-5x}$
$\Longrightarrow f(0) = -\cos(0) + 5\sin(0) + Ae^0 + B = -1 + A + B = 0 \Longrightarrow A + B = 1$
$\Longrightarrow f'(0) = \sin(0) + 5\cos(0) - A = 0 \Longrightarrow A = 5$ and $A + B = 1 \Longrightarrow B = 1 - 5 = -4$

Thus $f_g = -\cos(x) + 5\sin(x) + 5e^{-5x} - 4$

Exercise 3 (7pts)
Given the function $f(x)$, not nil defined by $f(x) = \frac{2e^x+3}{e^x-1}$ and its curve C_f
1.let's study limit of $f\ at -\infty\ and\ at +\infty$
At $+\infty$, we have

$\lim_{x\to-\infty}[f(x)] = \lim_{x\to-\infty}\left[\frac{2e^x+3}{e^x-1}\right] = -3, since\ as\ x \to -\infty, e^x \to 0$

At $+\infty$, we have

$\lim_{x\to+\infty}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{x\to+\infty}\left[\frac{2+\frac{3}{e^x}}{1-\frac{1}{e^x}}\right] = 2\ ; since\ as\ x \to +\infty, \frac{3}{e^x} \to 0\ and\ \frac{-1}{e^x} \to 0$

$\lim_{x\to-\infty} f(x) = -3\ and\ \lim_{x\to+\infty} f(x) = 2$

2.let's study the limits of f as x turns to zero by inferior value 0^- and by superior values 0^+

At 0^-, we have

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left[\frac{2e^x+3}{e^x-1}\right]$ let $x = \frac{1}{X}$ $then\ as\ x \to 0^-;\ X \to -\infty\ and \frac{1}{X} \to 0$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{X\to-\infty}\left[\frac{2e^{\frac{1}{X}}+3}{e^{\frac{1}{X}}-1}\right] = -3$

At 0^+, we have $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{X\to+\infty}\left[\frac{2+\frac{3}{e^{\frac{1}{X}}}}{1-\frac{1}{e^{\frac{1}{X}}}}\right] = +\infty$

3)let's deduce the asymptotes of C_f

4)let's deduce the derivative of f and study the variation of f.

$*$ $\forall x \in \mathbb{R}$; f is derivable$\Longrightarrow f'(x) = \left[\frac{2e^x+1}{e^x-1}\right]' = \frac{-3e^x}{(e^x-1)^2}$

Thus $\forall x \in \mathbb{R}, f'(x) = \frac{-3e^x}{(e^x-1)^2}$

$*$Variation of f.

$\forall x \in]-\infty; 0[,\ (e^x-1)^2 > 0\ and -3e^x < 0 \Longrightarrow \frac{-3e^x}{(e^x-1)^2} = f'(x) < 0$ …….(v)

$\forall x \in]0;\ +\infty[, (e^x-1)^2 > 0\ and -3e^x < 0 \Longrightarrow \frac{-3e^x}{(e^x-1)^2} = f'(x) < 0$ …..(vi)

$Therefor, \forall x \in \mathbb{R} - \{0\}; f'(x) < 0$ and it's strictly decreasing on $\mathbb{R} - \{0\}$

Exercise 4 (4pts)

There are four affirmations let's say which of them is true or false, given that $f(x) = \ln\left[\frac{2x+1}{x-1}\right]$

1. $f\ is\ defined\ on\ ]1;\ +\infty[$, **true** it's defined on $]-\infty; 0[$ and on $]1;\ +\infty[$.
2. $f'(x) = -\frac{1}{(x-1)^2}\ln\left(\frac{2x+1}{x-1}\right)$, ***false***, $f'(x) = \frac{2x-3}{(2x+1)(x-1)}$
3. The line $x = 1$ is an asymptote to the curve of f. **false.**
4. The curve of f admet a horizontal asymptote. **True**

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Exercise I (7mks)

A)Let f be a function defined on $]0;\ +\infty[$ by $f(x) = x - 2 + \frac{1}{2}\ln x$.

1)a-let's calculate the limits of f at 0 and at +

$At\ 0, we\ have, \lim_{x\to0} f(x) = \lim_{x\to0}\left[x - 2 + \frac{1}{2}\ln x\right] = -\infty$

At $+\infty$, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 2 + \frac{1}{2}\ln x\right] = +\infty$.

b)let's calculate $f'(x)$ and give the table of variation of f.

$\forall x \in]0;\ +\infty[, f\ is\ differentiable \Longrightarrow f'(x) = 1 + \frac{1}{2x} = \frac{2x+1}{2x}$

$\forall x \in]0;\ +\infty[, f'(x) = \frac{2x+1}{2x}$

We know from above that $x > 0 \Longrightarrow \forall x \in]0;\ +\infty[, \frac{2x+1}{2x} > 0$

Table of variation

$\forall x \in]0;\ +\infty[\ , f'(x) > 0, it's\ strickly\ increasing\ on\ ]0;\ +\infty[$

x		0 $\qquad$ $\frac{-1}{2}$ $\qquad$ $+\infty$
$f'(x)$		$+$
$f(x)$		$-\infty \nearrow 0 \nearrow +\infty$

4)a) let's show that the equation $f(x) = 0$, has a unique solution denoted by α on $]0;\ +\infty[$

x	1.75	1.74	1.73	1.72
$f(x)$	0.029	0.016	0.004	-0.001

f is defined and continuous and also strictly increasing in the interval $]0;\ +\infty[$
$f(]0;\ +\infty[) =]\lim_{x\to 0} f(x);\ \lim_{x\to+\infty} f(x)[=]-\infty;\ +\infty[$ and $0 \in]-\infty;\ +\infty[$,
Therefore

$f(x) = 0$ *has a unique solution* α *on* $]0;\ +\infty[$

Let's give the value of $\alpha\ at 10^{-2}\ near$.
We have $f(1) = 1 - 2 + \frac{1}{2}\ln 1 = -1$ and $f(2) = 2 - 2 + \frac{1}{2}\ln 2 = 0.35$

$f(1).f(2) < 0 \Longrightarrow \alpha \in]1;2[$,

$\frac{1+2}{2} = 1.5$

Also $f(1.5) = 1.5 - 2 + \frac{1}{2}\ln 1.5 = -0.29$

$f(2).f(1.5) < 0, \Longrightarrow \alpha \in]1;2[$

$\frac{1.5+2}{2} = 1.75 \Longrightarrow f(1.75) = 1.75 - 2 + \frac{1}{2}\ln 1.75 = 0.029$

$f(1.5).f(1.75) < 0 \Longrightarrow \alpha \in]1.5, 1.75[$

$\frac{1.5+1.75}{2} = 1.625 \Rightarrow f(1.625) = 1.625 - 2 + \frac{1}{2}\ln 1.625 = -0.13$

$f(1.75).f(1.625) < 0 \Longrightarrow \alpha \in]1.625; 1.75[$

. .
. .
. .

$f(1.72).f(1.73) < 0, \Longrightarrow \alpha \in]1.72; 1.73[\ since\ [1.73 - 1.72] = 10^{-2}$

B)let's g be a function defined on $[0;\ +\infty[$ by $g(x) = -\frac{7}{8}x^2 + x - \frac{1}{4}x^2\ln x, \forall x > 0, g(0) = 0$

1) a) let's study the continuity and the differentiation of $g\ in\ 0$.

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\left[-\frac{7}{8}x^2 + x - \frac{1}{4}x^2\ln x\right] = 0 = g(0)$ thus

$g(0) = \lim_{x\to 0^+} g(x) = 0$, it's continuous at 0

$\lim_{x\to 0^+}\left[\frac{g(x)-g\prime(0)}{x-0}\right] = \lim_{x\to 0^+}\left[-\frac{7}{8}x + 1 - \frac{1}{4}x\ln x\right] = 1$

Since $\lim_{x\to 0^+}\left[\frac{g(x)-g'(0)}{x-0}\right]\exists!$, we conclude that g is differentiable at 0

2)a) Let's calculate $g'(x) and\ verify\ that\ g'(x) = xf\left(\frac{1}{x}\right), \forall x > 0$

From above $g(x)$ is differentiable, thus $g'(x) = \left(-\frac{7}{8}x^2 + x - \frac{1}{4}x^2\ln x\right)'$

$$= -\frac{7}{4}x + 1 - \frac{1}{2}x(\ln x) - \frac{1}{4}x$$

$$= x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$$

Thus $g'(x) = x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$,

$\forall x > 0$

x	0		k		$+\infty$
$g'(x)$		$+$		$-$	
$g(x)$	0	$\nearrow$	$g(k)$	$\searrow$	$-\infty$

b)Let's deduce the sign of $g'(x)$ and draw the variation table.

$\forall x \in]0; k], g'(x) > 0\ it's\ strctly\ increasing\ on\ ]0; k]$

$\forall x \in [k;\ +\infty[, g'(x) < 0, it's\ strictly\ decreasing\ on\ [k;\ +\infty[$

4)let's give the equations of tangent to the curve of g at the point of $x - coordinate\ 0\ and\ 1$

At the point, $x = 0$, we have

$y_0 = g(0)(x - 0) + g(0) = x$

At the point, $x = 1$, we have

$y_1 = g(1)(x - 1) + g(1) = -x + 1 + \frac{1}{8} = -x + \frac{9}{8}$, thus equations are;

$$y_0 = x\ and\ y_1 = -x + \frac{9}{8}$$

$*$ let 's plot $c\ and\ the\ tangents$

Exercise II

a)Let's evaluate the following,s

1) $\int_0^{\sqrt{3}} x^2 \arctan(x)dx$.

$Let \begin{cases} u = \arctan(x) \\ v' = x^2 \end{cases} \Rightarrow \begin{cases} u' = \frac{1}{1+x^2} \\ v = \frac{1}{3}x^3 \end{cases}$ Using integration by parts we have,

$$\int_0^{\sqrt{3}} x^2 \arctan(x)\,dx = \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{2}} \frac{x^3}{1+x^2}dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{3}} \left[x - \frac{x}{1+x^2}\right]dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]^{\sqrt{3}} - \frac{1}{3}\left[\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(\sqrt{3}\right)^3 \arctan\left(\sqrt{3}\right) + \frac{3}{2} - \ln 2$$

$= \frac{\sqrt{3}\pi}{3} + \frac{3}{2} - \ln 2$, thus we have

$$\int_0^{\sqrt{3}} x^2 \arctan(x \quad \frac{\pi\sqrt{3}}{3} + \frac{3}{2} - \ln 2$$

2) $\int_0^1 \frac{1}{(1+x^2)^2}dx$

Let, $x = tg(t)$, for $x = 0, t = 0\ and\ for\ x = 1, t = \frac{\pi}{4}$

$$dx = \frac{1}{cos^2(t)}dt \Rightarrow \int_0^1 \frac{1}{(1+x^2)^2}dx = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{cos^{2(t)}}}{(1+(tg)^2)^2}dt$$

$$= \int_0^{\frac{\pi}{4}} cos^2(t)dt = \int_0^{\frac{\pi}{4}} \frac{1+cos2t}{2}dt = \frac{1}{2}\left[t+\frac{1}{2}sint\right]_0^{\frac{\pi}{4}} = \frac{\pi+2}{8}$$

,
Thus

$$\boxed{\int_0^1 \frac{1}{(1+x^2)^2}dx = \frac{\pi+2}{8}}$$

Exercise III

b)Let's solve the following differential equations.

$y' + y = 2\cos(x) + (x+1)e^{-x}$…………….$E$

∗ let's determine the characteristic equation of y.

We have, $y' + y = 0 \Longrightarrow \frac{y'}{y} = -1 \Longrightarrow \int \frac{1}{y}dy = \int(-1)dx \Longrightarrow \ln y_c = -x + k \Longrightarrow y_c = e^t e^{-x} = ke^{-x}$ $k \in \mathbb{R}$.

∗ let's determine the particular solution of

Let $y_c = A(x)e^{-x} \Longrightarrow {y'}_c = A'(x)e^{-x} - A(x)e^{-x}$. Substituting y_c *and* ${y'}_c$ into E, we have

$A'(x)e^{-x} - A(x)e^{-x} + A(x)e^{-x} = 2\cos(x) + (1+x)e^{-x}$

$\Longrightarrow A'(x)e^{-x} = 2\cos(x) + (x+1)e^{-x}$ ….(E)

Multiplying all through by e^x we have $A'(x) = 2e^x\cos(x) + (1+x)$. Integrating one has

$\int A'(x)dx = \int[2e^x\cos(x) + (1+x)]dx \Longrightarrow A(x) = 2\int e^x\cos(x)\,dx + x + \frac{x^2}{2}$ …………E'

From E', we have $\int e^x\cos(x)\,dx = e^x\cos(x) + \int e^x\sin(x)\,dx$

$$= e^x[\cos(x)+\sin(x)] - \int e^x\cos(x)\,dx + k$$

$$\Longrightarrow 2\int e^x\cos(x)\,dx = e^x[\cos(x)+\sin(x)] + k \Longrightarrow \int e^x\cos(x)\,dx = \frac{e^x}{2}[\cos(x)+\sin(x)] + c \;...E''$$

Substituting E'' into E', we have

$A(x) = e^x[\cos(x)+\sin(x)] + x + \frac{x^2}{2} + c, c \in \mathbb{R}$. Thus general

Solution is $y = e^x[\cos(x)+\sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, k \text{ and } c \in \mathbb{R}$.

$$\boxed{y = e^x[\cos(x)+\sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, \qquad \forall k, c \in \mathbb{R}.}$$