JULY 2011

Exercise I (7mks)

A) Let f be a function defined on $]0;\ +\infty[$ by $f(x) = x - 2 + \frac{1}{2}\ln x$.

1) a-let's calculate the limits of f at 0 and at $+\infty$

$At\ 0, we\ have, \lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x - 2 + \frac{1}{2}\ln x\right] = -\infty$

At $+\infty$, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 2 + \frac{1}{2}\ln x\right] = +\infty$

Thus

$$\boxed{\lim_{x\to 0} f(x) = -\infty\ and\ \lim_{x\to+\infty} f(x) = +\infty}$$

b) Let's calculate $f'(x)$ and give the table of variation of f.

$\forall x \in\ ]0;\ +\infty[, f\ is\ differentiable \Longrightarrow f'(x) = 1 + \frac{1}{2x} = \frac{2x+1}{2x}$

Then $\boxed{\forall x \in\ ]0;\ +\infty[, f'(x) = \frac{2x+1}{2x}}$

We know from above that $x > 0 \Longrightarrow \forall x \in\ ]0;\ +\infty[, \frac{2x+1}{2x} > 0$

Then $\boxed{\forall x \in\ ]0;\ +\infty[\,, f'(x) > 0, it's\ strictly\ increasing\ on\ ]0;\ +\infty[}$

Table of variation

x	0	$\frac{-1}{2}$	$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$ $\nearrow$	0 $\nearrow$	$+\infty$

2) a) let's show that the equation $f(x) = 0$, has a unique solution denoted by α on $]0;\ +\infty[$

x	1.75	1.74	1.73	1.72
$f(x)$	0.029	0.016	0.004	-0.001

f is defined and continuous and also strictly increasing in the interval $]0;\ +\infty[$

$f(]0;\ +\infty[) =]\lim_{x\to 0} f(x);\ \lim_{x\to+\infty} f(x)[=]-\infty;\ +\infty[$ and $0 \in]-\infty;\ +\infty[$,

Therefore $\boxed{f(x) = 0\ has\ a\ unique\ solution\ \alpha\ on\ ]0;\ +\infty[}$

3.Let's give the value of $\alpha\ at 10^{-2}\ near$.

We have $f(1) = 1 - 2 + \frac{1}{2}\ln 1 = -1$ and $f(2) = 2 - 2 + \frac{1}{2}\ln 2 = 0.35$

$f(1).f(2) < 0 \Longrightarrow \alpha \in]1; 2[$,

$\frac{1+2}{2} = 1.5$

Also $f(1.5) = 1.5 - 2 + \frac{1}{2}\ln 1.5 = -0.29$

$f(2).f(1.5) < 0, \Longrightarrow \alpha \in]1; 2[$

$\frac{1.5+2}{2} = 1.75 \Longrightarrow f(1.75) = 1.75 - 2 + \frac{1}{2}\ln 1.75 = 0.029$

$f(1.5).f(1.75) < 0 \Longrightarrow \alpha \in]1.5, 1.75[$

$\frac{1.5+1.75}{2} = 1.625 \Rightarrow f(1.625) = 1.625 - 2 + \frac{1}{2}\ln 1.625 = -0.13$

$f(1.75).f(1.625) < 0 \Longrightarrow \alpha \in]1.625; 1.75[$

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$f(1.72).f(1.73) < 0, \Longrightarrow \alpha \in]1.72; 1.73[$ $since$ $[1.73 - 1.72] = 10^{-2}$

B) Let g be a function defined on $[0;\ +\infty[$ by $g(x) = -\frac{7}{8}x^2 + x - \frac{1}{4}x^2 \ln x, \forall x > 0, g(0) = 0$

1) a) let's study the continuity and the differentiation of g in 0.

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}\left[-\frac{7}{8}x^2 + x - \frac{1}{4}x^2 \ln x\right] = 0 = g(0)$$

Thus

$\boxed{g(0) = \lim_{x\to 0^+} g(x) = 0}$, it's continuous at 0

$$\lim_{x\to 0^+}\left[\frac{g(x)-g\prime(0)}{x-0}\right] = \lim_{x\to 0^+}\left[-\frac{7}{8}x + 1 - \frac{1}{4}x\ln x\right] = 1$$

Since $\lim_{x\to 0^+}\left[\frac{g(x)-g'(0)}{x-0}\right]$ ∃!, we conclude that g is differentiable at 0

2) a) Let's calculate $g'(x) and\ verify\ that\ g'(x) = xf\left(\frac{1}{x}\right), \forall x > 0$

From above $g(x)$ is differentiable, thus $g'(x) = \left(-\frac{7}{8}x^2 + x - \frac{1}{4}x^2 \ln x\right)'$

$$= -\frac{7}{4}x + 1 - \frac{1}{2}x(\ln x) - \frac{1}{4}x$$

$$= x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$$

Thus $g'(x) = x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$, $\forall x > 0$ ■

b)Let's deduce the sign of $g'(x)$ and draw the variation table.

$\forall x \in]0; k], g'(x) > 0\ it's\ strictly\ increasing\ on\ ]0; k]$

$\forall x \in [k;\ +\infty[, g'(x) < 0, it's\ strictly\ decreasing\ on\ [k;\ +\infty[$

x	0	k	$+\infty$
$g'(x)$	$+$		$-$

$g(x)$	0	$g(k)$	$-\infty$

3)Let's equations of tangent to the curve of g at the point of $x - coordinate$ 0 and 1

At the point, $x = 0$, we have

$$y_0 = g(0)(x-0) + g(0) = x$$

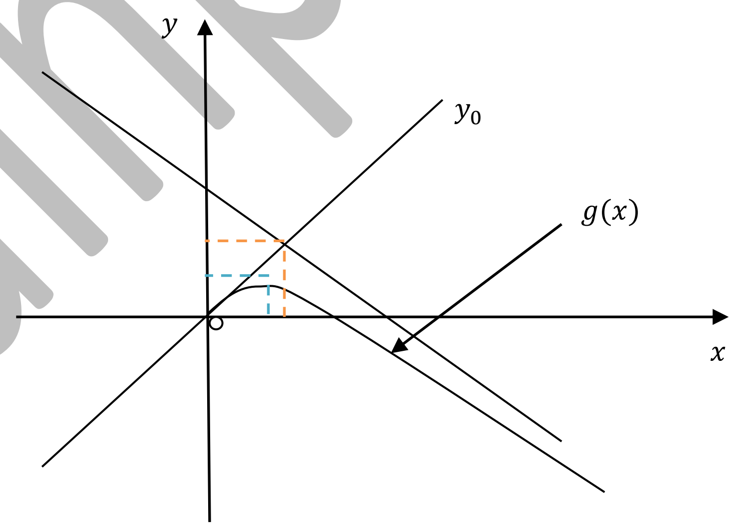
At the point, $x = 1$, we have

$y_1 = g(1)(x-1) + g(1) = -x + 1 + \frac{1}{8} = -x + \frac{9}{8}$,

thus equations are;

$$\boxed{y_0 = x \ and \ y_1 = -x + \frac{9}{8}}$$

$*$ Let 's plot c and the $tangents$



Exercise II

Let's evaluate the following,

$\int_0^{\sqrt{3}} x^2 \arctan(x)dx.$ *Let* $\begin{cases} u = \arctan(x) \\ v' = x^2 \end{cases} \Longrightarrow \begin{cases} u' = \frac{1}{1+x^2} \\ v = \frac{1}{3}x^3 \end{cases}$

Using integration by parts we have,

$$\int_0^{\sqrt{3}} x^2 \arctan(x)\, dx = \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{2}} \frac{x^3}{1+x^2} dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{3}} \left[x - \frac{x}{1+x^2}\right] dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\left[\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(\sqrt{3}\right)^3 \arctan\left(\sqrt{3}\right) + \frac{3}{2} - \ln 2$$

$$= \frac{\sqrt{3}\pi}{3} + \frac{3}{2} - \ln 2,$$

Thus we have

$$\boxed{\int_0^{\sqrt{3}} x^2 \arctan(x)\, dx = \frac{\pi\sqrt{3}}{3} + \frac{3}{2} - \ln 2}$$

1. $\int_0^1 \frac{1}{(1+x^2)^2} dx$

Let, $x = tg(t)$, for $x = 0, t = 0$ *and for* $x = 1, t = \frac{\pi}{4}$

$dx = \frac{1}{cos^2(t)} dt \Longrightarrow \int_0^1 \frac{1}{(1+x^2)^2} dx = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{cos^2(t)}}{(1+(tg)^2)^2} dt$

$\int_0^{\frac{\pi}{4}} cos^2(t)dt = \int_0^{\frac{\pi}{4}} \frac{1+cos2t}{2} dt = \frac{1}{2}\left[t + \frac{1}{2}sint\right]_0^{\frac{\pi}{4}} = \frac{\pi+2}{8}$, Thus

$$\boxed{\int_0^1 \frac{1}{(1+x^2)^2} dx = \frac{\pi+2}{8}}$$

Exercise III

Let's solve the following differential equations.

1. $y' + y = 2\cos(x) + (x+1)e^{-x}$……………$E$

$*$ let's determine the characteristic equation of E

We have, $y' + y = 0 \Longrightarrow \frac{y'}{y} = -1$

$\Longrightarrow \int \frac{1}{y} dy = \int (-1)dx \Longrightarrow \ln y_c - x + k \Longrightarrow y_c = e^t e^{-x} = ke^{-x}\ k \in \mathbb{R}$.

$*$ let's determine the particular solution of

Let $y_c = A(x)e^{-x} \Longrightarrow y'_c = A'(x)e^{-x} - A(x)e^{-x}$.

Substituting y_c *and* y'_c into E, we have

$A'(x)e^{-x} - A(x)e^{-x} + A(x)e^{-x} = 2\cos(x) + (1+x)e^{-x}$

$A'(x)e^{-x} = 2\cos(x) + (x+1)e^{-x}$ ….E'

Multiplying all through by e^x we have $A'(x) = 2e^x\cos(x) + (1+x)$. Integrating one has

$\int A'(x)dx = \int [2e^x\cos(x) + (1+x)]dx \Longrightarrow A(x) = 2\int e^x \cos(x)\,dx + x + \frac{x^2}{2}$ …….E'

From E', we have $\int e^x \cos(x)\,dx = e^x\cos(x) + \int e^x \sin(x)\,dx$

$$\int \mathrm{e^x}\ \cos(\mathrm{x})\ \mathrm{dx} = e^x[\cos(x) + \sin(x)] - \int e^x \cos(x)\,dx + k \Longrightarrow 2\int e^x\cos(x)\,dx$$

$$= e^x[\cos(x) + \sin(x)] + k$$

$$\Longrightarrow \int e^x \cos(x)\,dx = \frac{e^x}{2}[\cos(x) + \sin(x)] + c \ldots E''$$

Substituting E'' into E', $A(x) = e^x[\cos(x) + \sin(x)] + x + \frac{x^2}{2} + c, c \in \mathbb{R}$.

Thus general Solution is $y_p = e^x[\cos(x) + \sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, k$ *and* $c \in \mathbb{R}$.

$$\boxed{y_{p+c} = e^x[\cos(x) + \sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, \qquad \forall k, c \in \mathbb{R}.}$$

Exercise IV

Given numerical sequences u_n *and* v_n defined by

$u_0 = 2, \forall n \epsilon \mathbb{N}$, and $v_n = \frac{2}{u_n}$ *and* $u_{n+1} = \frac{u_n + v_n}{2}$

1) Let's calculate $v_0; u_1;\ v_1;\ u_2;\ v_2$; given the answers in the form of non-reducible fraction.

when $n = 0$, we have $v_0 = \frac{2}{u_0} = \frac{2}{2} = 1$ and $u_{0+1} = \frac{u_0+v_0}{2} = \frac{2+1}{2} = 1\frac{1}{2}$

when $n = 1$, we have $v_1 = \frac{2}{u_1} = \frac{2}{\frac{3}{2}} = \frac{4}{3} = 1\frac{1}{3}$ and $u_{1+1} = \frac{u_1+v_1}{2} = \frac{\frac{3}{2}+\frac{4}{3}}{2} = \frac{\frac{17}{6}}{2} = \frac{17}{12} = 1\frac{5}{12}$

when $n = 2$, we have $v_2 = \frac{2}{u_2} = \frac{2}{\frac{17}{12}} = \frac{2x12}{17} = \frac{24}{17} = 1\frac{7}{17}$

$$\boxed{v_0 = 1;\ u_1 = 1\frac{1}{2}\ ;\ v_1 = 1\frac{1}{3};\ u_2 = 1\frac{5}{12};\ v_2 = 1\frac{7}{17}}$$

1) Let's show that these sequences are bounded above by 2 and bounded below by 1.ie

$*$ let's show that $\forall n \in \mathbb{N}, 1 \leq u_n \leq 2$

Suppose $n = 0\ then\ u_0 = 2 \implies 1 \leq u_0 \leq 2$ which is true.

Assume the results to be true for $= (k > 0) \in \mathbb{N}$, then $1 \leq u_k \leq 2$ which is true.

Let's show the results to be true for $n = (k+1) \in \mathbb{N}$, then $1 \leq u_{k+1} \leq 2$

$\implies 1 + v_k \leq v_k + u_k \leq 2 + v_k \implies \frac{1+v_k}{2} \leq \frac{u_k+v_k}{2} \leq \frac{2+v_k}{2} \implies \frac{1+\frac{2}{u_k}}{2} \leq u_{k+1} \leq \frac{2+\frac{2}{u_k}}{2}$

From $1 \leq u_k \leq 2 \implies \frac{1}{2} \leq u_k \leq 1 \implies 1 \leq \frac{2}{u_k} \leq 2 \implies 2 \leq \frac{2}{u_k} \leq 1$

$1 \leq \frac{2}{u_k} \implies 2 \leq 1 + \frac{2}{u_k} \implies 1 \leq \frac{1+\frac{2}{u_k}}{2}$ ... ... (i).

Therefore

$\frac{2}{u_k} \leq 2 \implies 2 + \frac{2}{u_k} \leq 4 \implies \frac{2+\frac{2}{u_k}}{2} \leq 2$ ... ... (ii). Combining $(i) and\ (ii)$ we have,

$1 \leq \frac{1+\frac{2}{u_k}}{2} \leq u_{k+1} \leq \frac{2+\frac{2}{u_k}}{2} \leq 2 \implies 1 \leq u_n \leq 2$. Thus $\boxed{\forall n \in \mathbb{N}; 1 \leq u_n \leq 2}$

$*$ Let's show that $\forall n \in \mathbb{N}; 1 \leq v_n \leq 2$

Suppose $n_0 = 0, v_0 = 1\ and\ 1 \leq v_0 \leq 2$, which is true. Assume the results to be true for $n_0 = k > 0 \forall n \in \mathbb{N}$, we have $1 \leq v_k \leq 2$, which is also true. And let's show the results to be true for $1 \leq v_{k+1} \leq 2$

We have $1 \leq \frac{2}{u_k} \implies 1 \leq \frac{2}{u_{k+1}} \leq 2 \implies 1 \leq v_{k+1} \leq 2$, which is also true as $v_{k+1} = \frac{2}{u_{k+1}}$.

Hence $\boxed{\forall n \in \mathbb{N}; 1 \leq v_n \leq 2}$

2) Let's show that $\forall n \in \mathbb{N}, u_{n+1} - v_{n+1} = \frac{(u_n-v_n)^2-8}{2(u_n+v_n)}$

We have $u_{n+1} = \frac{u_n+v_n}{2}\ and\ v_{n+1} = \frac{2}{u_{n+1}} = \frac{4}{u_n+v_n}$

Then, $u_{n+1}-v_{n+1}=\frac{u_n+v_n}{2}-\frac{4}{v_n+u_n}=\frac{(u_n-v_n)^2}{2(u_n+v_n)}$, therefore

$$\boxed{\forall n\in\mathbb{N}, u_{n+1}-v_{n+1}=\frac{(u_n-v_n)^2-8}{2(u_n-v_n)}}$$ ■

3) Let's show that $\forall n\in\mathbb{N}, u_n \geq v_n$

From 2) above suppose we have $2(u_n-v_n)>0, (u_n-v_n)^2-8\geq 0$

$\Longrightarrow \frac{(u_n-v_n)^2-8}{2(u_n-v_n)}\geq 0$ thus $u_n\geq v_n$ $\boxed{\forall n\in\mathbb{N};\ u_n\geq v_n}$

4) $*$Let's show that (u_n) is a decreasing sequence.

We have $u_{n+1}+u_n=\frac{u_n+v_n}{2}-u_n=\frac{v_n-u_n}{2}$, but we have

$u_n\geq v_n \Longrightarrow v_n-u_n\leq 0\Longrightarrow \frac{v_n-u_n}{2}\leq 0\Longrightarrow u_{n+1}-u_n\leq 0\ and\ u_{n+1}\leq u_n$

$\boxed{\textit{since } \forall n\in\mathbb{N}, u_{n+1}\leq u_n \text{ we conclude that } u_n \text{ is a decreasing sequence}}$

$**$Let's show that v_n is an increasing sequence

We have $v_{n+1}-v_n=\frac{2}{v_{n+1}}-v_n=\frac{2}{\frac{v_n+u_n}{2}}-v_n=\frac{4}{u_n+v_n}-v_n=\frac{4-v_n(v_n-u_n)}{u_n+v_n}$

But $2\leq u_n+v_n\leq 4\Longrightarrow -4v_n\leq -v_n(u_n+v_n)\leq -2v_n\Longrightarrow -v_n(u_n+v_n)\leq -2v_n$

$\Longrightarrow 4-v_n(u_n+v_n)\leq 4-2v_n\Longrightarrow 4-v_n(u_n+v_n)\geq 0$

$\Longrightarrow \frac{4-v_n(u_n+v_v)}{(u_n+v_n)}\geq 0\Longrightarrow v_{n+1}-v_n\geq 0\Longrightarrow v_{n+1}\geq v_n$

$\boxed{\text{Since } \forall n\in\mathbb{N}, v_{n+1}\geq v_n \text{, we conclude that } v_n \text{ is an increasing sequence}}$ ■

6)Let's show that $\forall n\in\mathbb{N};\ u_n-v_n\leq 1$ and deduce that $(u_n-v_n)^2\leq(u_n-v_n)$

We had $1\leq u_n\leq 2$…(i) and $1\leq v_n\leq 2\Longrightarrow -2\leq -v_n\leq -1$ ... (ii)

$(i)+(ii)\Longrightarrow -1\leq u_n-v_n\leq 1 \ \Longrightarrow u_n-v_n\leq 1$ …(iii)multiplying through by u_n-v_n we have $(u_n-v_n)(u_n-v_n)\leq(u_n-v_n)\Longrightarrow(u_n-v_n)^2\leq(u_n-v_n)$ …(iv)

$\boxed{(iii)\ and(iv)\Longrightarrow\forall n\in\mathbb{N};\ u_u-v_n\leq 1\ and\ (u_n-v_n)^2\leq(u_n-v_n)}$ ■

9) a) let's show that

$\forall n\in\mathbb{N}, u_{n+1}-v_{n+1}=\frac{1}{4}(u_n-v_n)$,

We have had $2\leq u_n+v_n\leq 4\Longrightarrow 4\leq 2(u_n+v_n)\leq 8$, multiplying by 2.

Dividing by 8, we have $\frac{1}{8} \leq \frac{1}{\frac{1}{2}(u_n - v_n)} \leq \frac{1}{4}$ considering the RHS and multiplying it by $(u_n - v_n)$ we have $\frac{(u_n - v_n)}{\frac{1}{2}(u_n + v_n)} \leq \frac{1(u_n - v_n)}{4}$ but $u_n - v_n \geq 0 \Longrightarrow u_{n+1} - v_{n+1} \leq \frac{1(u_n - v_n)}{4}$

Thus $$\boxed{\forall n \in \mathbb{N}; we\ have\ u_{n+1} - v_{n+1} \leq \frac{(u_n - v_n)}{4}}$$

b) Let's show that $\forall n \in \mathbb{N};\ u_n - v_n \leq \frac{1}{4^n}$

As k varies from 0 $to\ n$, we have $u_1 - v_1 \leq \frac{(u_0 - v_0)}{4}$, $u_2 - v_2 \leq \frac{1(u_1 - v_1)}{4}$

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$u_{n-1} - v_{n-1} \leq \frac{1}{4}(u_{n-2} - v_{n-2})$

$u_n - v_n \leq \frac{(u_{n-1} - v_{n-1})}{4}$

Multiplying each member by its self we've $u_n - v_n \leq [\left(\frac{1}{4}\right)^n (u_0 - v_0) = \left(\frac{1}{4}\right)^n (2 - 1)]$

$$\boxed{\text{Therefore } \forall n \in \mathbb{N};\ u_n - v_n \leq \frac{1}{4^n}}$$

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8)let's show that u_n and v_n converge to the same limits

We have $\lim_{n\to+\infty}(u_n - v_n) = \lim_{n\to+\infty}(\frac{1}{4^n}) \Longrightarrow \lim_{n\to+\infty} u_n - \lim_{n\to+\infty} v_n = 0$ and we get $\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n$

$$\boxed{\forall n \in \mathbb{N};\ \lim_{n\to+\infty} u_n = \lim_{n\to+\infty} v_n}$$

■