SOLUTIONS

JULY 2009

Exercise I (4mks)

We are given two sequences (U_n) defined by $U_0 = 2, U_1 = 3\ and\ U_n = \frac{4U_{n-1}-U_{n-2}}{3}$, and (V_n) defined by $V_n = U_n - U_{n-1}$, where $n\epsilon\mathbb{N}^*$

a) Let's show that (V_n) is a geometrical sequence and determine the first term and the common ratio.

$*$We have $U_n = \frac{4U_{n-1}-U_{n-2}}{3} \dots\dots\dots\dots\dots (i), V_n = U_n - U_{n-1}\ \ , n\epsilon N$…………..(ii).

From equation (ii) we have $V_{n-1} = U_{n-1} - U_{n-2}$ ……………………. (iii)

Substituting equation (i) into equation (ii) gives $V_n = \frac{4U_{n-1}-U_{n-2}}{3} - U_{n-1} = \frac{U_{n-1}-U_{n-2}}{3} = \frac{1}{3}V_{n-1} \dots$ (iv) From (iii). Therefore, $V_n = \frac{1}{3}V_{n-1}$

$*$ For $n = 1, and\ substituting\ in\ (ii) gives\ V_1 = U_1 - U_0 = 3 - 2 = 1$

$*$ From (iv) the common ratio is $\frac{1}{3}$

Therefore

$$From\ V_n = \frac{1}{3}V_{n-1} shows\ that V_n is\ a\ gp. first\ term\ 1, common\ ratio\ \frac{1}{3}$$

b) Let's calculate the general term in terms of n.

Generally we have $V_n = ar^n, a = V_1 = first\ term, r = \frac{1}{3} common\ ratio\ and\ substituting\ from\ (a) gives$

$V_n = ar^n = V_1 r^n = 1.\left(\frac{1}{3}\right)^n = 3^{-n}$.

Therefore

$$General\ term\ V_n = 3^{-n}, \forall n\epsilon N$$

c) Let's calculate $S_n = V_1 + V_2 + V_3 + \cdots + V_n\ in\ terms\ of\ n$

We have $S_n = V_1 + V_2 + \cdots V_{n-1} + V_n = (U_1 - U_0) + (U_2 - U_1) + \cdots + U_n - U_{n-1}$

$$= U_n - U_0 = \frac{a(1-r^n)}{1-r} = \frac{V_1(1-r^n)}{1-r} = \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}} = \frac{1}{2}(3-3^{1-n})$$

Therefore,

$S_n = \frac{1}{2}(3-3^{1-n})$, $\forall n \epsilon N$

d) Let's show that the sequence (U_n), converge and specify it's limit.

$S_n = U_n - U_0 = \frac{1}{2}(3-3^{1-n})$, hence $U_n = 2 + \frac{1}{2}(3-3^{1-n}) = \frac{1}{2}(7-3^{1-n})$

$U_{n+1} = \frac{1}{2}\left(7+3^{1-(1+n)}\right) = \frac{1}{2}(7+3^{-n})$, $U_{n+1} - U_n = 3^{-n} > 0$…. (e), $\therefore U_n$ *is strictly increasing.*

$* \lim_{n\to\infty} U_n = \lim_{n\to\infty}\left[\frac{1}{2}(7+3^{1-n})\right] = \frac{7}{2}$ ……………… (f)

since U_n *is strickly increasing and* $\lim_{n\to+\infty} U_n = \frac{7}{2}$,

U_n *converges to* $\frac{7}{2}$

Therefore,

Exercise II (3mks)

1) Let's evaluate, $\int_0^{\cosh^{-1}(2)} \frac{\tanh x}{1+\cosh x}dx$, given that $u = 1+\cosh x$ leaving our answer in natural logarithm.

$\tanh x = \frac{\sinh x}{\cosh x}$.

$$Let\ U(x) = 1+\cosh(x) = 1 + \frac{e^x+e^{-x}}{2}, then\ U(0) = 1 + \frac{e^0+e^{-0}}{2} = 2\ and$$

$$U(\cosh^{-1}(x)) = 1+\cosh(\cosh^{-1}(x)) = 3\ ,but\ U-1 = \cosh(x)\ and$$

$$dU = \sinh(x)dx$$

Thus $$H = \int_0^{\cosh^{-1}x} \frac{\tanh x}{1+\cosh x}dx = \int_2^3 \frac{\sinh x}{\cosh x(1+\cosh x)}dx$$

$$= \int_2^3 \frac{dU}{(U-1)U} = \int_2^3\left(\frac{1}{U-1} - \frac{1}{U}\right)dU \text{ by partial fractions}$$

$$H = [\ln|U-1|]_2^3 - [|\ln U|]_2^3$$

$$= (\ln 2 - \ln 1) - (\ln 3 - \ln 2)$$

$$= \ln\frac{4}{3}$$

$$H = \ln\frac{4}{3}$$

2) Let's prove that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ and hence or otherwise show that $\int_0^{\frac{1}{2}} \tanh^{-1} x dx = \frac{1}{4}\ln\left[\frac{27}{16}\right]$

∗Let $y = \tanh^{-1}(x)$ *then we have* $x = \tanh y$ *and* $sech^2(y)\frac{dy}{dx} = 1$ but

$sech^2(y) = 1 - tanh^2(y)$ Therefore $\frac{dy}{dx} = \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{sech^2(y)} = \frac{1}{1-tanh^2(y)} = \frac{1}{1-x^2}$

∗ By using integration by parts, let's show that $\int_0^{\frac{1}{2}} \tanh^{-1}(x)dx = \frac{1}{4}\ln\left[\frac{27}{16}\right]$

We let $\begin{cases} v(x) = \tanh^{-1}(x) \\ u'(x) = 1 \end{cases} \Rightarrow \begin{cases} v'(x) = \frac{1}{(1-x^2)} \\ u(x) \quad = x \end{cases}$

$$\Rightarrow \int_0^{\frac{1}{2}} \tanh^{-1}(x)dx = [x\tanh^{-1}(x)]_0^{\frac{1}{2}} - \frac{1}{2}\int_0^{\frac{1}{2}}\left[-\frac{2x}{(1-x^2)}\right]dx$$

$$= \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\right) + \frac{1}{2}[\ln|1-x^2|]_0^{\frac{1}{2}} = \frac{1}{2}\tanh^{-1}\left(\frac{1}{2}\right) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{2}.\frac{1}{2}\ln\left[\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right] + \frac{1}{2}\ln\left(\frac{3}{4}\right) = \frac{1}{4}\ln(3) + \frac{1}{2}\ln\left(\frac{3}{4}\right)$$

$$= \frac{1}{4}\left[\ln 3 + 2\ln\left(\frac{3}{4}\right)\right] = \frac{1}{4}\left[\ln 3 + \ln\left(\frac{9}{16}\right)\right] = \frac{1}{4}\ln\left[\frac{27}{16}\right]$$

Therefore

■ $$\frac{d(\tanh^{-1}(x))}{dx} = \frac{1}{(1-x^2)} \quad and \quad \int_0^{\frac{1}{2}} \tanh^{-1}(x)dx = \frac{1}{4}\ln\left[\frac{27}{16}\right]$$

Exercise III (4mks)

Let's determine the Cartesian equation of the plane, τ passing through;

$a)$ $P(-2; 6; 7)$ *and has a normal vector* $\vec{n}(0; 3; 0)$:

Let $M(x; y; z)$ be a point on the plane, then $\overrightarrow{PM} = \begin{bmatrix} x+2 \\ y-6 \\ z-7 \end{bmatrix}, \vec{n} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $\overrightarrow{PM}$ is perpendicular to τ hence.

Implying that $\overrightarrow{PM}.\vec{n} = \vec{0} \leftrightarrow \begin{bmatrix} x+2 \\ y-6 \\ z-7 \end{bmatrix}.\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 3(y-6) = 0.$

Thus y = 6

b) $P(-6;10;16)$ And its perpendicular to the right hand side of AB.

Given, $\begin{bmatrix}1\\0\\5\end{bmatrix}$; $B\begin{bmatrix}3\\-3\\2\end{bmatrix}$; $AB=\begin{bmatrix}2\\-3\\-3\end{bmatrix}$; $Let M\begin{bmatrix}x\\y\\z\end{bmatrix}$; $then\ \overrightarrow{PM}=\begin{bmatrix}x+6\\y-10\\z-16\end{bmatrix}$, τ is perpendicular to AB hence to $\overrightarrow{PM}$.

Thus $PM.AB=0 \Rightarrow \begin{bmatrix}x+6\\y-6\\z-16\end{bmatrix}.\begin{bmatrix}2\\-3\\-3\end{bmatrix}=0$ giving us , $2(x+6)-3(y-10)-3(z-16)=0$

$Thus$ Cartesian Equation of the plane is $2x-3y-3z+90=0$

Exercise iv (4mks)

1) Let's find the square root $a+bi$ of the complex number $Z_1=5+12i$, $\sqrt{z}$

$$\begin{cases}a^2+b^2=|z_1|\\ a^2-b^2=Re(z_1)\\ 2ab=Im(z_1)\end{cases} \Leftrightarrow \begin{cases}a^2+b^2=13\ \ldots\ldots\ldots(i)\\ a^2-b^2=5\ \ldots\ldots\ldots\ldots(ii)\\ 2ab=11\ \ldots\ldots\ldots\ldots\ldots(iii)\end{cases}$$

$(i)+(ii)=2a^2=18 \Rightarrow a=\pm3$

$(i)-(ii)=2b^2=8 \Leftrightarrow b=\pm2$ $\quad Thus$ $z_A=3+2i\ and z_B=-3-2i$

2) Let's find the modulus and argument of the complex number $z_2=\frac{(1+i)^2}{(-1+i)^4}$

$$z_2=(1+i)^2(-1+i)^{-4}=\sqrt{2}\left(cos\frac{\pi}{4}+sin\frac{\pi}{4}\right)^2\sqrt{2}\left(cos\frac{\pi}{4}-sin\frac{\pi}{4}\right)^{-4}$$

$$=\left[\sqrt{2}\left(cos\frac{\pi}{4}+sin\frac{\pi}{4}\right)\right]^2\left[\sqrt{2}\left(cos\frac{\pi}{4}-sin\frac{\pi}{4}\right)\right]^{-4}$$

$$=2\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)\frac{1}{4}(cos\pi+sin\pi)=\frac{1}{2}i(-1)$$

Therefore $z_2=-\frac{1}{2}i$

3) Let's calculate $z_3z_3^*$ and $\frac{z_3}{z_3^*}$ if $z_3=1+i\sqrt{3}$

$*$We have $z_3z_3^*=(1+i\sqrt{3})(1-i\sqrt{3})=1+3=4$

$*$We have $\frac{z_3}{z_3^*}=\frac{(1+i\sqrt{3})}{(1-i\sqrt{3})}=\frac{(1+i\sqrt{3})(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})}=-\frac{1}{2}+i\frac{\sqrt{3}}{4}$,

. Therefore, $$\boxed{z_3 z_3^* = 4 \ and \ \frac{z_3}{z_3^*} = -\frac{1}{2} + i\frac{\sqrt{3}}{4}}$$

Exercise v (3mks)

Let's calculate the following quantities

a) $\int \frac{3x+7}{x^2+x+1} dx$

$\partial \ and \ \varphi$ Therefore $\int \frac{3x+5}{x^2+x+1} dx = \int \frac{\frac{3}{2}(2x+1)+\frac{7}{2}}{x^2+x+1} dx = \int \frac{\frac{3}{2}(2x+1)}{x^2+x+1} dx + \int \frac{\frac{7}{2}}{x^2+x+1} dx$

$$= \frac{3}{2}\ln(x^2+x+1) + \frac{7}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C, C\epsilon IR.$$

Thus, $$\boxed{\int \frac{3x+5}{x^2+x+1} dx = \frac{3}{2}\ln(x^2+x+1) + \frac{7}{2}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C, C\epsilon IR}$$

b) $\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx$

Let $3x+7 = \gamma(-2x^2+x+1)' + \rho \ (\gamma, \rho \epsilon \mathbb{R})$

$= \gamma(-4x+1) + \rho$, therefore $\begin{cases} \rho = \frac{-3}{4} \\ \gamma = \frac{31}{4} \end{cases}$ by equating coefficients and solving for γ & ρ.

$\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx = \int \frac{\frac{-3}{4}(-4x+1)+\frac{31}{4}}{\sqrt{-2x^2+x+1}} dx = \frac{-3}{4}\int \frac{(-4x+1)}{\sqrt{-2x^2+x+1}} dx + \frac{31}{4}\int \frac{1}{\sqrt{-2x^2+x+1}} dx = -\frac{3}{4}A + \frac{31}{4}B$

Let $U^2 = -2x^2+x+1$, $then \ 2UdU = (-4x+1)dx$ substituting in A gives us

$\frac{-3}{4}A = -\frac{3}{4}\int \frac{2UdU}{\sqrt{U^2}} = -\frac{3}{4}\int 2dU = -\frac{3}{2}U + K' = -\frac{3}{2}\sqrt{-2x^2+x+1} + K'$

$\frac{31}{4}B = \frac{31}{4}\int \frac{1}{\sqrt{-2x^2+x+1}} dx = \frac{31}{4\sqrt{2}}\int \frac{1}{\sqrt{\frac{9}{16}-\left(x-\frac{1}{4}\right)^2}} dx$, by completing the square

$$= \frac{31}{4\sqrt{2}}\int \frac{1}{\sqrt{\left[\frac{3}{4}\right]^2 + \left[x-\frac{1}{4}\right]^2}} dx = \frac{31}{4\sqrt{2}} arcsin\left(\frac{4x-1}{3}\right) + K''$$

$$\boxed{\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx = \frac{-3}{4}\sqrt{-2x^2+x+1} + \frac{31}{4\sqrt{2}}\sin^{-1}\left(\frac{4x-1}{3}\right) + K, K = K' + K''\epsilon\mathbb{R}}$$

Exercise 6

Let's consider the probability density function $''p.d.f.''$ defined given by

$$f(x) = \begin{cases} cx^3(1-x^2), if \ 0 \le x \le 1 \\ 0, \quad elsewhere \end{cases}$$

a)Let’s calculate the value of the constant c

A continuous random variable$X$ will be a probability density function iff

$f(x) \geq 0, and\ c\int_{-\infty}^{+\infty} f(x)dx = 1$,

$f(x) \geq 0 \Leftrightarrow cx^3(1-x^2) \geq 0 \Rightarrow c \geq 0, x^3 \geq 0\ and\ (1-x^2) \geq 0$

$\int_0^1 cx^3(1-x^2)dx = \int_0^1 cx^3\,dx - \int_0^1 \frac{1}{6}cx^6 dx = 1$, since $x \in [0;1]$

$$= \frac{c}{4}[x^4]_0^1 - [x]_0^1 = 1 \Leftrightarrow \frac{c}{4} - \frac{c}{6} = \frac{c}{12} = 1 and\ c = 12$$

$$\boxed{c = 12}$$

b) Let’s find the mean and variance of X

$*$Mean, $(X) of\ X$,

we have $E(X) = \int_0^1 xf(x)dx = \int_0^1 x[12x^3(1-x^2)]dx = 12\int_0^1 (x^4 - x^6)dx$

$$= \frac{12}{5}[x^5]_0^1 - \frac{12}{7}[x^7]_0^1 = \frac{12}{5} - \frac{12}{7} = \frac{24}{35}$$

Therefore $\boxed{E(X) = \frac{24}{35}}$

$* The\ value\ of\ Var(X)$,

we have $Var(X) = E(x^2) - [E(X)]^2$

$E(x^2) = \int_0^1 x^2 f(x)dx = \int_0^1 x^2.12x^3[1-x^2]dx = \int_0^1 12(x^5 - x^7)dx = 12\left[\frac{x^6}{6} - \frac{x^8}{8}\right]_0^1 = \frac{12}{6} - \frac{12}{8} = \frac{1}{2}$

$\Leftrightarrow E(x^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{24}{35}\right)^2 = \frac{73}{2230}$,

Therefore, $\boxed{Var(X) = \frac{37}{2230}}$

c) Let’s determine the mode of X

$from\ p.d.f., We\ have\ f(x) = [12x^3 - 12x^5] = 0 \Leftrightarrow f'(x) = 12x^2(3-5x^2) = 0$

$\therefore Either\ 12x^2 = 0 \Rightarrow x = 0, or\ 3 - 5x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{3}{5}}$

but $x \in [0;1] \Leftrightarrow x > 0 \Rightarrow x = \sqrt{\frac{3}{5}}$ $\therefore$ $\boxed{\text{The mode } x = \sqrt{\frac{3}{5}}}$

d).Let's show that $6m^4 - 4m^6 + 1 = 0, given\ m\ as\ a\ median\ of\ X$

If X is a continuous random variable in the range $x_0 \leq x \leq x_k\ k \in \mathbb{R}, then\ we\ have$ $the\ median, m\ as\ the\ value\ of\ X\ if\ P(x_0 \leq x \leq x_k) = \frac{1}{2}$

$i.e. \int_{x_0}^{x_k} f(x)dx = \frac{1}{2}$

$\Longrightarrow \int_0^m (12x^3 - 12x^5)dx = \frac{1}{2}$

$$\Longrightarrow (3x^4 - 2x^6)_0^m = \frac{1}{2} \Longrightarrow 6m^4 - 4m^6 = 1\ and\ then\ we\ have$$

$$\boxed{6x^4 - 4x^6 - 1 = 0}$$ ■