SEPTEMBER 2012

Exercise I (1.5+1+1.5=4)

Given that $\int_0^{\frac{\pi}{2}} cos^n\,(x)dx$

1-Let's calculate I_0 , $I_1 and\ I_2$.

We have $\int_0^{\frac{\pi}{2}} cos^n\,(x)dx$, so when $n = 0,1,2,$ *we* have respectively

$I_0 = \int_0^{\frac{\pi}{2}} cos^0(x)dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$I_1 = \int_0^{\frac{\pi}{2}} cos^1(x)dx = [\sin(x)]_0^{\frac{\pi}{2}} = sin\frac{\pi}{2} - sin0$$

$$I_2 = \int_0^{\frac{\pi}{2}} cos^2\,(x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos(2x))\,dx$$

$I_2 = \frac{1}{2}\left[x + \frac{1}{2}sin2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}sin2\left(\frac{\pi}{2}\right) - 0 - sin0\right] = \frac{\pi}{4}$ Therefore we have

$$\boxed{I_0 = \frac{\pi}{2},\quad I_1 = 1\ and\ I_2 = \frac{\pi}{4}}$$

b) Let's show that $\forall n \geq 2, one\ has\ nI_n = (n-1)I_{n-2}$

We suppose $U(x) = cos^{n-1}(x)\ and\ V'(x) = \cos(x). Also\ cos^n(x) = cos^{n-1}(x)\cos(x)$

$$\begin{cases} U'(x) = -(n-1)cos^{n-2}(x)\sin(x) \\ V(x) = \sin(x) \end{cases}$$

Using integration by parts we have $I_n = \mathrm{V}(x).\mathrm{U(x)} = \int_0^{\frac{\pi}{2}} \mathrm{U'(x)\,V(x)dx} \ldots \ldots \mathrm{(i)}$, substituting $U(x), U'(x) and\ V'(x), V(x)\ \ into\ (i)$ gives

$I_n = [\sin(x)\cos^{n-1}(x)]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin(x)\cos^{n-2}(x)\sin(x)\,dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} sin^2\,(x)cos^{n-2}(x)dx = (n-1)\int_0^{\frac{\pi}{2}}(1 - cos^2(x))\,cos^{n-2}(x)dx$

$= (n-1)\int_0^{\frac{\pi}{2}} cos^{n-2}(x)dx - (n-1)\int_0^{\frac{\pi}{2}} cos^n\,(x)dx = (n-1)I_{n-2} - (n-1)I_n$

$\Leftrightarrow I_n = (n-1)I_{n-2} - (n-1)I_n \Leftrightarrow I_n + (n-1)I_n = nI_n = (n-1)I_{n-2}$ Thus

$$\boxed{nI_n = (n-1)I_{n-2}}$$

c) Let's deduce the value of $I_n\ \forall n \geq 1$

From (b) we have $nI_n = (n-1)I_{n-2} \Leftrightarrow \frac{I_n}{I_{n-2}} = \frac{n-1}{n} = 1 - \frac{1}{n} \Leftrightarrow n = \frac{I_{n-2}-I_n}{I_{n-2}}$.

Thus we have $$\boxed{\frac{I_{n-2}-I_n}{I_{n-2}} = n\quad \forall n \geq 1}$$

Exercise II (4mks)

1)Let's study the continuity of the following function

$$f(x)=\begin{cases} x^2\ if\ x\leq 0 \\ x\ \ if\ 0<x<2 \\ 4-x\ \ if\ x\geq 2 \end{cases} \longrightarrow \begin{cases} g(x) if\ x\leq 0 \\ h(x)\ if\ 0<x<2 \\ i(x)\ \ if\ x\geq 2 \end{cases}$$

$*$We have $\lim_{x\to 0^-} x^2=0=g(0)$ hence is continuous $\forall x\leq 0$

$*$Let $x=1\in]0;2[$ then we have $\lim_{x\to 1} x=1=h(1)$ thus $h(x) is\ continuous\ \forall x\in]0;2[$

$*$Lastly, we have $\lim_{x\to 2^+} 4-x=2=i(2)$, hence $i(x)\ is\ continuous\ \forall x\geq 2$

Since$f(x) is$ continuous$\forall x\leq 2;\ \forall x\in]0;2[\ and\ \forall\geq 2\ we\ conclude\ it's\ continuous\ \forall x\in\mathbb{R}$

2) Let's show $g(x)$ is continuous and derivable and it's derivative $g'(x)$is continuous

We've $g(x)=\begin{cases} x^2\ \ if\ x\leq 0 \\ 0\ \ if\ x=0 \\ \cos(x)-1\ if x>0 \end{cases} \Leftrightarrow \begin{cases} h(x) if\ x<0 \\ g(x)\ if\ x=0 \\ i(x)\ if\ x>0 \end{cases}$

$*$ We've $\lim_{x\to 0^-} x^2=0=\lim_{x\to 0^+}(cos(x)-1)\ hence\ g(x) is\ cotinuous\ \dots\dots\dots\dots.(i)$

$$g'(x)=\begin{cases} -\frac{1}{x^2}e^{\frac{1}{x}}\ if\ x<0 \\ 0\ \ if\ x=0 \\ -\sin(x)\ if\ x>0 \end{cases}$$

$*$We've $\lim_{x\to 0^-}\frac{h(x)-h'(0)}{x-0}=\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}+\frac{e^{\frac{1}{x}}}{x^2}}{x}=\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}(1+\frac{1}{x^2})}{x}$

Let $X=\frac{1}{x}\Leftrightarrow As\ x\longrightarrow 0^-, X\longrightarrow -\infty$ Substituting X in the above we've

$$\lim_{x\to 0^-}\frac{e^{\frac{1}{x}}(1+\frac{1}{x^2})}{1/\frac{1}{x}}=\lim_{X\to -\infty}\frac{e^X(1+X^2)}{\frac{1}{X}}=\lim_{X\to -\infty}Xe^X(1+X^2)=0$$

Also from $g'(x)$ we've $\lim_{x\to o^+}\sin(x)=0$

Therefore $g(x)$ differentiable $\dots\dots\dots(ii)$

$*\ Determination\ of\ continuity\ of\ g'(x)$

$from\ (i) limits\ of\ g(x)\ exist\ and\ from\ (ii)\ g(x)\ is\ respectively$

$continuous\ and\ derivable\ and\ g'(x)$
$is\ continuous\ from\ (iii)$

$We\ have\ lim_{x\to 0^-}(-\frac{1}{x^2}e^{\frac{1}{x}}) = 0 = lim_{x\to o^+} sin(x)\ \ from\ g'(x)\ above\ \ldots\ldots\ldots\ldots.(iii)$

Exercise III (3mks)

Given a discrete random variable X with the probability distribution with $E(X) = 2$, with distribution table below.

$(X = x)$	0	1	2	3	4
$P(X = x)$	0.1	p	0.25	q	0.05

a) Let's find the values of $p\ and\ q$

We know that for a discrete random variable, $X, \sum_{i=0}^{4} x_i P(X = x_i) = 1$. Hence we have

$\sum_{i=0}^{4} x_i\, P(X = x_i) = 0x0.0.1 + 1xp + 2x0.25 + 3xq + 4x0.05 = E(X) = 2$

$\Leftrightarrow 1xp + 2x0.25 + 3xq + 4x0.05 = 2 \Leftrightarrow p + 3q = 1.3 \ldots\ldots.(i)$

Also $\sum_{i=0}^{4} P(X = x_i) = 0.1 + +p + 0.25 + q + 0.05 = 1 \Leftrightarrow p + q = 0.6 \ldots\ldots\ldots.(ii)$

$\begin{cases} p + 3q = 1.3 \ldots\ldots\ldots.(i) \\ p + q = 0.6 \ldots\ldots\ldots\ldots(ii) \end{cases} \Leftrightarrow \begin{cases} p = 0.35 \\ q = 0.25 \end{cases}$ By solving $(i) and\ (ii)$

Therefore we have

$p = 0.25\ and\ q = 0.35$

b) Let's find $Var(X)$.We know that, $Var(X) = E(X^2) - E^2(X) = E(X^2) - \mu^2$, where $E(X^2) = \sum_0^4 x^2\, P(X = x) \Leftrightarrow E(X^2) = 0^2x0.1 + 1^2x0.25 + 2^2x0.25 + 3^2x0.35 + 4^2x0.05 = 5.00$

$Var(X) = E(X^2) - E^2(X) = (5.00) - (2.02)^2 = 5.00 - 4.81 = 0.19.$

Therefore we $Var(X) = 0.19$ have

c) Let's calculate the average $E(Y)$ and variance $Var(Y)$ of $Y = 5X + 4$

∗Average of Y

$E(Y) = E(5X + 4) = 5E(X) + 4 = 5x2.02 + 4 = 10.10 + 4 = 14.10 \ldots\ldots(1)$

$*$ Variance of Y

$$Var(Y) = Var(5X+4) = Var(5X) + Var(4)$$
$$= 5^2 Var(X) + 0 = 25x0.19 = 4.75 \ldots.(2)$$

Therefore $$(1) and\ (2) \Longrightarrow E(Y) = 14.10\ and\ Var(Y) = 4.75$$

Exercise III (9mks).

Part 1 (6mks).

Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{3(x-1)^3}{3x^2+1}$ and let C be its curve

1) Let's that there exists a single triplet $(a; b; c)$ that one will determine such as for real x

$f(x) = ax + b + \frac{cx}{3x^2+1}$

We have $f(x) = \frac{3(x-1)^3}{3x^2+1} = \frac{3x^3-9x^2+9x-3}{3x^2+1}$ by expansion and using long division method one has

$\frac{3x^3-9x^2+9x-3}{3x^2+1} = x - 3 + \frac{8x}{3x^2+1} \equiv ax + b + \frac{cx}{3x^2+1}$………..(I')

Therefore we have

$$From\ (I')\ that\ f(x) = x - 3 + \frac{8x}{1+3x^2} \Longrightarrow a = 1, b = -3\ and\ c = 8$$

2) Let's determine the limits of $f\ in\ \pm\infty$

At $+\infty$, $\text{Lim}_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 3 + \frac{8x}{3x^2+1}\right]$

$$= \lim_{x\to+\infty} x\left[1 - \frac{3}{x} + \frac{8}{3x+\frac{1}{x}}\right] = +\infty\ since\ \frac{1}{x} \to 0. as\ x \to +\infty$$

At $-\infty$, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x - 3 + \frac{8x}{3x^2+1}\right) = \lim_{x\to-\infty} x\left[1 - \frac{3}{x} + \frac{8}{3x+\frac{1}{x}}\right] = -\infty$ Therefore

$$\lim_{x\to+\infty} f(x) = +\infty\ and\ \lim_{x\to-\infty} f(x) = -\infty$$

3) Let's show that f is differentiable and calculate its derivative

$*$ f is made up of three parts $x - 3, 8x\ and\ 3x^2 + 1$ all differentiable on $\mathbb{R}\ or\ on\ ]-\infty;\ +\infty[$ therefore $f\ is\ itself\ differentiable\ on\ \mathbb{R}\ or\ on\ ]-\infty;\ +\infty[\ \ldots\ldots\ldots\ldots (a)$

Hence $\forall x \in\ ]-\infty;\ +\infty[\ or\ \mathbb{R}$, we have $f'(x) = \left[x - 3 + \frac{8}{3x^2+1}\right]' = 1 + \frac{24x^2-48x+8}{(3x^2+1)^2} \ldots\ldots..(b)$.

Hence

$$\boxed{From\ (a), f(x)\ is\ differentiable\ on\ \mathbb{R}\ and\ from\ (b), f'(x) = 1 + \frac{24x^2 - 48x + 8}{(3x^2+1)^2}}$$

4) Let's draw the table of variation of f.

$\forall x \epsilon\ ]-\infty; 0]\ \ f'(x) > 0, hence\ positive$ and $\forall x \epsilon [0;\ +\infty[\ f'(x) > 0\ hence\ increasing.$

There no turning points since x has no real value(s). Intercepts: when $x0\ subsituting\ x = 0\ into\ y = f(x)\ gives\ y = \frac{3(0-1)^3}{3(0)^2+1} = -3.$ hence $(0;\ -3)$.

When $f(x) = 0\ we've\ 0 = \frac{3(x-1)^3}{3x^2+1} \Leftrightarrow x = 1, hence\ (1; 0)$

x	$-\infty$		0		$+\infty$
$f'(x)$		+ +		+ +	+
$f(x)$	$-\infty$	↗	0	↗	$+\infty$

5) Let's show that the curve (C) has the line $(D): y = x - 3,$ as oblique asymptote $\lim_{x\to\pm\infty}[f(x) - y] = \lim_{x\to\pm\infty}\left[x - 3 + \frac{8x}{1+3x^2} - (x-3)\right] = \lim_{x\to\pm\infty}\left[\frac{8x}{3x^2+1}\right] = 0 \ldots.(k)$

Therefore

$$\boxed{The\ line\ y = x - 3\ is\ an\ oblique\ asymptote\ to\ (C)\ \ from\ (k)}$$ ■

6) Let's study the relative positions

We have $f(x) - y = \frac{8x}{3x^2+1}$ we know that $\forall x \in \mathbb{R}, x^2 > 0\ and\ 3x^2 + 1 > 0.$ therefore the sign of $f(x) - y\ depends\ on\ that\ of\ 8x,\ \ and$ we have that

$$\forall x \in]-\infty; 0], 8x < 0 \; and \; \frac{8x}{3x^2+1} < 0 \Longrightarrow f(x) - y < 0, thus\ negative, \forall x \in]-\infty; 0] \ldots\ldots\ldots..(1)$$

$$\forall x \in [0; +\infty[\; 8x > 0 \; and \; \frac{8x}{3x^2+1} > 0, hence\ f(x) - y, thus\ positive \ldots.... \quad (2)$$

$From\ (1)\ f(x) < y\ hence f(x) is\ below\ y\ \forall x \in]-\infty; 0]\ and\ from\ (2)\ f(x) > y,$ $hence\ f(x) is\ above\ y, \forall x \in [0; +\infty[$

7) Let's give the equation of tangent (T) to (C) at the point of x –coordinate 0. And trace (T), $(D) and$ (C).

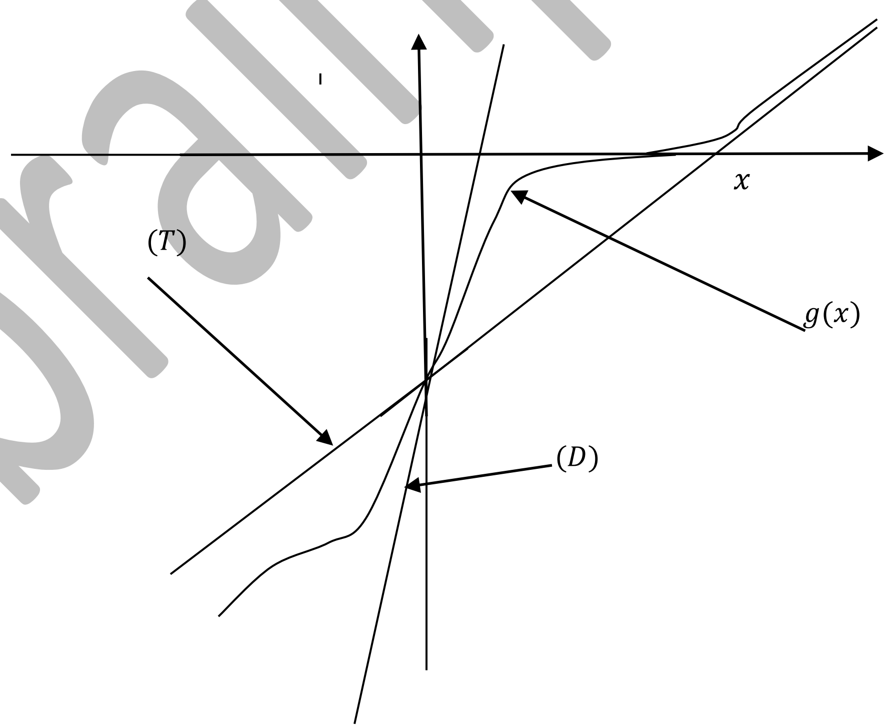
An equation of tangent to a curve at x_0 is given by $y - f(x_0) = f'(x_0)[x - x_0];\ x_0 = 0$

Substituting for $x_0 = 0\ in\ the\ above\ we've$

$$y - \left[\frac{3(x_0-1)^3}{3x_0{}^2+1}\right] = \left[1 + \frac{24x_0 - 48x_0 + 8}{(3x_0^2+1)^2}\right](x - x_0) = y + 3 = 9x;\ \ x_0 = 0 \Leftrightarrow y = 9x - 3$$

$equation\ of\ tangent\ is\ y = 9x - 3$

Let's trace $(D), (T)\ and\ (C)$.



8) Let's show that the curve(C) has the centre of symmetry.

9) Let's show that the equation $f(x) = 1$ has a single solution in $\mathbb{R}$ denoted as α.

10) Let's give the approximate value of $\alpha\ to\ 10^{-2}$ nearby excess.

Part II (3mks)

Given the function f defined by on $\mathbb{R}$ by $g(x) = \frac{3(\sin(x)-1)^3}{3sin^2(x)+1}$

1) Let's show that g is differentiable on $\mathbb{R}$ and calculate $g'(x)$

$*\ g$ consists of functions $3(\sin(x) - 1)^3\ and\ sin^2(x) + 1$ which are all differentiable on $\mathbb{R}$ thus g is differentiable on $\mathbb{R}$ …………..(i)

$*\ g'(x) = \left(\frac{(\sin(x)-1)^3}{\sin^2(x)+1}\right)' = \frac{2\cos(x)[sin^2(x)\cos(x)-3sin^2(x)-15\sin(x)+9\cos(x)]}{(sin^2(x)+1)^2}$ $\forall x \in \mathbb{R}$. Using quotient rule.

Therefore $g'(x) = [9cosx(sinx - 1)^2(\sin^2 x + 2sinx + 1)]/(3\sin^2 x + 1)^2$

$= \frac{9cosx(sinx-1)^2(sinx+1)^2}{(3sinx+1)^2}$.

Therefore

$$g'(x) = \frac{9cosx(sinx - 1)^2(sinx + 1)^2}{(3sinx + 1)^2}$$

2. Let's draw the table of variation of

At turning points $g'(x) = 0 \Longrightarrow \frac{9cosx(sinx-1)^2(sinx+1)^2}{(3sinx+1)^2} = 0\ and\ get$

$cosx = 0 \Longrightarrow x = (2n + 1)\pi, \sin x = 1, -1. We\ have\ x \in \left\{\frac{-\pi}{2}; \frac{\pi}{2}\right\}\ in\ the\ range\ [-\pi, \pi]$

$g(-\pi) = -3\ and\ g(\pi) = -3$ and $g(0) = -3$

x	$-\pi$	$\frac{-\pi}{2}$		$\frac{\pi}{2}$	π
$g'(x)$	$-$	0	$+$	0	$-$
$g(x)$	↘	-6	↗	0	↘

3) Let's draw or plot a new drawing the representative curve of g.