JULY 2010

Exercise 1

(u_n) is the sequence defined by $u_0 = 3;\ u_{n+1} = \frac{4}{u_n}$, …….(a)

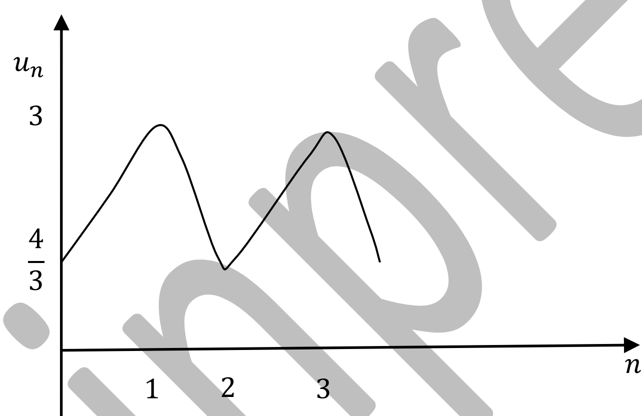
1) a) Let's calculate $u_1, u_2, u_3\ and\ u_4$.

For $n = 0$ from (a) we have $u_1 = \frac{4}{u_0} = \frac{4}{3}$, For $n = 1$, we have $u_2 = \frac{4}{u_1} = 4\left(\frac{3}{4}\right) = 3$,

For $n = 2$, we have $u_3 = \frac{4}{u_2} = 4\left(\frac{1}{3}\right) = \frac{4}{3}$, For $n = 3$, we have $u_4 = \frac{4}{u_3} = 4\left(\frac{3}{4}\right) = 3$.

b. Let's conjecture about the nature of the sequence u_n

The sequence is sinusoidal and takes 2 values; 3 for even values of n and $\frac{4}{3}$ for odd values of n.



2) Given that $v_n = \frac{-2+u_n}{2+u_n}$

a) Let's show that v_n is a geometric sequence and determine it.

We have $v_{n+1} = \frac{-2+u_{n+1}}{2+u_{n+1}} = \frac{-2+\frac{4}{u_n}}{2+\frac{4}{u_n}} = \frac{-2u_n+4}{2u_n+4} = \frac{-2(u_n-2)}{2(u_n+2)} = -v_n$

Thus $v_{n+1} = -v_n \iff \frac{v_{n+1}}{v_n}$ = -1. And $v_0 = \frac{-2+u_0}{2+u_0} = \frac{-2+3}{2+3} = \frac{1}{5}$ …….(b)

Thus $\boxed{v_n \text{ is a geometrical sequence with common ratio } \frac{v_{n+1}}{v_n} = -1 \text{ and first term } v_0 = \frac{1}{5}}$

b) Let's express v_n in terms of n and u_n in terms of n.

We have $v_n = \frac{-2+u_n}{2+u_n} \Leftrightarrow (2+u_n)v_n = -2+u_n \Leftrightarrow u_n = \frac{-2-2v_n}{v_n-1}$

Thus $u_n = \frac{-2-2\left[\frac{1}{5}(-1)^n\right]}{\frac{1}{5}(-1)^n-1} = \frac{-10-2(-1)^n}{(-1)^n-5}$. Implying $\boxed{v_n = \frac{-2+\frac{-10-2(-1)^n}{(-1)^n-5}}{2+\frac{-10-2(-1)^n}{(-1)^n-5}} \text{ and } u_n = \frac{-10-2(-1)^n}{(-1)^n-5}}$

c) Let's show that $\forall n \geq 2, u_{n+2} = u_n$ $and\ deduce$ u_{2009}

For $= 2, u_4 = 3, u_2 = 3 \Longrightarrow u_4 = u_2$, thus $u_{2+2} = u_2$

Let $k \geq 2 \in \mathbb{N}$, and $p(k)$ the proposition: For $n \geq 2, u_{n+2} = u_2$

Suppose that $u_{k+2} = u_k$ and prove that $u_{k+3} = u_{k+1}$, we have

$$u_{k+3} = \frac{-10-2(-1)^{k+3}}{(-1)^{k+3}-5} = \frac{-10-2(-1)^{k+1}(-1)^2}{(-1)^{k+1}.(-1)^2-5} = \frac{-10-2(-1)^{k+1}}{(-1)^{k+1}-5} = u_{k+1}$$

Let's deduce u_{2009} . We've $u_{2009} = u_{2007+2} = u_{2007}$ and $u_{2007} = \frac{-10-2(-1)^{2007}}{(-1)^{2007}-5} = \frac{4}{3}$

$\boxed{\text{We conclude that } \forall n \geq 2, u_{n+2} = u_n \text{ and } u_{2009} = \frac{4}{3}}$

Exercise 2 (3mks)

1) a) Let's verified that $\forall x \in \mathbb{R}, cos2x = 2cos^2x - 1$.

From definition $cos2x = cos^2x - sin^2x = cos^2x - 1 + cos^2x \Leftrightarrow cos2x = 2cos^2x - 1$

Thus $\boxed{cos2x = 2cos^2x - 1}$

b) Let's deduce the exact value of $cos2x$ given that $cosx = \frac{\sqrt{6}+\sqrt{2}}{4}$.

From (a) above $cos2x = 2cos^2x - 1 \Rightarrow cos2x = 2\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2 - 1 = \frac{2(8+2\sqrt{3})-16}{16} = \frac{4\sqrt{3}}{16} = \frac{\sqrt{3}}{4}$

Thus $\boxed{cos2x = \frac{\sqrt{3}}{4}}$

3) Let's solve inside $]0;\ \pi]$ the equation $cos2x = \frac{\sqrt{3}}{2}$

We have, $cos2x = \frac{\sqrt{3}}{2} \Leftrightarrow \begin{cases} cos2x = \cos\left(\frac{\pi}{6}\right) + 2k\pi \\ cos2x = \cos\left(\frac{\pi}{6}\right) + 2k\pi \end{cases}, k \in \mathbb{Z}$

$\Leftrightarrow x = \begin{cases} \frac{\pi}{12} + k\pi \\ -\frac{\pi}{12} + k\pi \end{cases}$ $, k \in \left]0; \frac{\pi}{2}\right] \in \mathbb{Z}$. For $k = 0, \Rightarrow x = \frac{\pi}{12}$ $, k = 1 \Rightarrow x = \frac{11\pi}{12} \notin \left]0; \frac{\pi}{2}\right]$

thus

$$\boxed{\text{The solution of } cos2x = \frac{\sqrt{3}}{2} \text{ is } x = \left\{\frac{\pi}{12}\right\}}$$

Exercise 3 (3mks)

a) Let's express the complex numbers, $\frac{z}{z'}$ and $z\left(cos\frac{\pi}{4} - sin\frac{\pi}{4}\right)$ in the form $a + ib$.

∗ Expression of $\frac{z}{z'}$ in the form $a + ib$. We have $\frac{z}{z'} = \frac{2-3i}{2+3i} = \frac{(2-3i)(2+3i)}{(2+3i)(2-3i)} = \frac{-5}{12} - \frac{12}{13}i$

∗ Expression of $z\left(cos\frac{\pi}{4} - isin\frac{\pi}{4}\right)$ in the form $a + ib$

$we\ have\ z\left(cos\frac{\pi}{4} - isin\frac{\pi}{4}\right) = (2-3i)\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i$, thus we have respectively that:

$$\boxed{\frac{z}{z'} = -\frac{\sqrt{2}}{2} - \frac{12}{13}i \ and \ z\left(cos\frac{\pi}{4} - isin\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{5\sqrt{2}}{2}i}$$

b) Let's calculate $\frac{d^2y}{dx^2}\ for\ t = \frac{\pi}{2}$ and that $x = a(t + sint), y = a(1 - cost)$

$\frac{dx}{dt} = [a(t + sint)]' = a(1 + cost)$, $\frac{dy}{dt} = [a(1 - cost)]' = asint$

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{a(sint)}{a(1+cost)} = \frac{sint}{1+cost}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{sint}{1+cost}\right)}{\frac{dx}{dt}} = \frac{1}{a(1+cost)}\left[\frac{cost(1+cost) - sint(-sint)}{(1+cost)^2}\right]$$

$$= \frac{cost + cos^2t + sin^2t}{(1+cost)^2} \cdot \frac{1}{a(1+cost)} = \frac{cost+1}{(1+cost)^2} \cdot \frac{1}{a(1+cost)}$$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{1}{a(1+cost)^2}$, when, $= \frac{\pi}{2}$, we have $\frac{d^2y}{dx^2}_{,t=\frac{\pi}{2}} = \frac{1}{a\left(1+cos\frac{\pi}{2}\right)^2} = \frac{1}{a}$.

Thus $$\boxed{\frac{d^2y}{dx^2} = \frac{1}{a}}$$

Exercise 4 (4mks)

Let's evaluate the following integrals

a) $\int_0^1 \frac{x}{(x+2)(x+3)}dx$.

We have $\frac{x}{(x+2)(x+3)} \equiv \frac{a}{x+2} + \frac{b}{x+3} \Longrightarrow \frac{a(x+3)+b(x+2)}{(x+2)(x+3)} \equiv \frac{x}{(x+2)(x+3)}$ multiplying all through by $(x+2)(x+3)$ we have $a(x+2) + b(x+3) \equiv x$

$\Longrightarrow (a+b)x + 2a + 3b \equiv x$, equating coefficients we have

$\begin{cases} a+b = 1 \ldots (i) \\ 2a+3b = 0 \ldots (ii) \end{cases}$ solving the two equations for a and b we have

From $(i), b = 1 - a$, putting in (ii) we have $2a + 3(1-a) = 0 \Longrightarrow a = 3$ and $b = 1 - 3 = -2$

$\Leftrightarrow \frac{x}{(x+2)(x+3)} \equiv \frac{3}{x+2} - \frac{2}{x+3}$

$\int_0^1 \frac{x}{(x+2)(x+3)} dx = \int_0^1 \left[\frac{3}{x+2} - \frac{2}{x+3}\right] dx = 3[\ln(x+2)]_0^1 - 2[\ln(x+3)]_0^1 = 3[\ln 4 - \ln 3] - 2[\ln 3 - \ln 2]$

$= -5\ln 3 + 8\ln 2$

Thus $\boxed{\int_0^1 \frac{x}{(x+2)(x+3)} dx = -5\ln 3 + 8\ln 2}$

$b)\ \int_0^1 xe^{x+2}\, dx.$

Let $= x \implies u' = 1$ *and* $v' = e^{x+2} \implies v = e^{x+2}$. Using integration by parts we have

$\int_0^1 xe^{x+2}\, dx = [xe^{x+2}]_0^1 - \int_0^1 e^{x+2} dx = [xe^{x+2}]_0^1 - [e^{x+2}]_0^1 = e^3 - e^3 + e^2$

Thus $\boxed{\int_0^1 xe^{x+2} dx = e^2}$

b) $\int_0^{\frac{\pi}{3}} sin^2 x dx.$

we have $\int_0^{\frac{\pi}{3}} sin^2 x dx = \int_0^{\frac{\pi}{3}} \frac{1-cos2x}{2} dx$, since $\frac{1-cos2x}{2} = sin^2 x$

$\Longrightarrow \int_0^{\frac{\pi}{3}} \frac{1-cos2x}{2} dx = \frac{1}{2}\left[\int_0^{\frac{\pi}{3}} 1dx - \int_0^{\frac{\pi}{3}} cos2xdx\right] = \frac{1}{2}\left[(x)_0^{\frac{\pi}{3}} - \frac{1}{2}(sin2x)_0^{\frac{\pi}{3}}\right] = A$

$\Longrightarrow A = \frac{1}{2}\left[\frac{\pi}{3} - \frac{1}{2} sin\frac{2\pi}{3}\right] = \frac{\pi}{6} - \frac{\sqrt{3}}{8}$.

Thus

$$\boxed{\int_0^{\frac{\pi}{3}} sin^2xdx = \frac{\pi}{6} - \frac{\sqrt{3}}{8}}$$

Exercise 5

Let's differentiate the following functions with respect to x.

a) $f(x) = e^{3x}(sinx - 3cosx)$.

$\forall x \in \mathbb{R},\ f(x)$ is differentiable, thus

$f'(x) = [e^{3x}(sinx - 3cosx)]' = 3e^{3x}(sinx - 3cosx) + e^{3x}(cosx + 3sinx)$

$= e^{3x}(6sinx - 8cosx)$.

Thus

$$\boxed{f'(x) = e^{3x}(6sinx - 8cosx)}$$

c) $g(x) = \frac{x^2}{1+x}$

$\forall x \in \mathbb{R} - \{-1\}$, $g(x)$ is differentiable, thus

$g'(x) = \left[\frac{x^2}{1+x}\right]' = \frac{2x(1+x)-x^2}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2}$, therefore

$$\boxed{\forall x \in \mathbb{R} - \{-1\}, g'(x) = \frac{x^2 + 2x}{(1 + x)^2}}$$

d) Given the relation,$12v = x^2\sqrt{3b^2 + x^2}$, where $b > 0$. Let's show that the maximum value of v as x varies is $\frac{b^3}{6}$

From $12v = x^2\sqrt{3b^2 + x^2} \Longrightarrow v = \frac{x^2}{12}\sqrt{3b^3 + x^2}$

$v'(x) = \frac{2x}{12}\left[\sqrt{3b^2 + x^2} + \frac{x^2}{2\sqrt{3b^3+x^2}}\right] = \frac{x}{4}\left[\frac{2b^2+x^2}{\sqrt{3b^2+x^2}}\right]$.

At maximum

Value $v, v'(x) = 0$. Thus, $\frac{x}{4}\left[\frac{2b^2+x^2}{\sqrt{3b^2+x^2}}\right] = 0 \Longrightarrow x = 0$ or $2b^2 + x^2 = 0$. From where $x^2 = -2b^2$. From $12v = x^2\sqrt{3b^2 + x^2}$ and $x^2 = -2b^2$

$\Longrightarrow 12v = x^2\sqrt{3b^2 - 2b^2} = x^2\sqrt{|b^2|}$. The maximum value of v is obtained when $x^2 > 0 \Longleftrightarrow$

$12v_{max} = 2b^2.b = 2b^3 \Longrightarrow v_{max} = \frac{2b^3}{12} = \frac{b^3}{6}$.

Thus

$$\boxed{v_{max} = \frac{b^3}{6}}$$