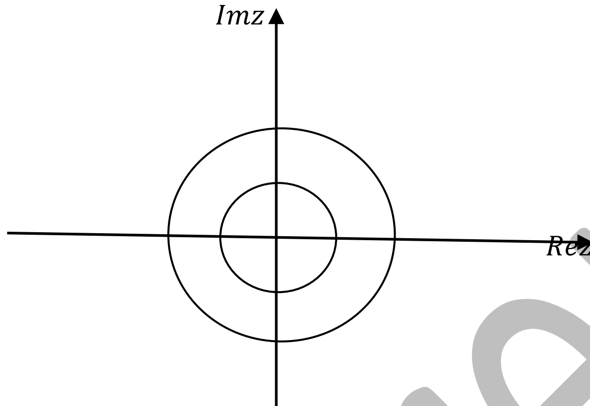
JULY 2009

Exercise 1 (5pts)

1. Let's draw the circles with centre 0 and radius $1\ and\ 2$. indicate the points A; B; C; D, with affixes $\sqrt{3}+i$, $\sqrt{3}-i$, and $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$



2) Given the rotation on R with centre 0 and an angle $\frac{\pi}{3}$ and the translation T of a vector with affix I:

a) Let's determine affixes $z_{A'}$ and $z_{B'}$, respective images of points A and B by rotation R.

For $Z_{A'}$ we have $(Z_{A'}-0)=e^{i\frac{\pi}{3}}\left[\left(\sqrt{3}+i\right)-0\right]=e^{i\frac{\pi}{3}}\left(\sqrt{3}+i\right)=2e^{i\frac{\pi}{3}}e^{i\frac{\pi}{3}}=2e^{i\frac{2\pi}{3}}$

For $Z_{B'}$ we have $(Z_{B'}-0)=e^{i\frac{\pi}{3}}\left[\left(\sqrt{3}-i\right)-0\right]=e^{i\frac{\pi}{3}}\left(\sqrt{3}-i\right)=2$. Thus we have

$$Z_{A'}=e^{i\frac{\pi}{3}}\left(\sqrt{3}+i\right)=2e^{i\frac{2\pi}{3}}\ and\ Z_{B'}=e^{i\frac{\pi}{3}}\left(\sqrt{3}-i\right)=2$$

b) Let's find or determine the affix $Z_{D'}$ of point D' image of point D by the translation R.

We have $Z_{D'}=Z_D+a, a=1,\ affix\ of\ vector \Rightarrow Z_{D'}=\left[-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right]+1=\frac{1}{2}+\frac{\sqrt{3}i}{2}$.

Thus we have

$$Z_{D'}=\frac{1}{2}+\frac{\sqrt{3}i}{2}$$

c)Let's indicate the points $A', B', and\ D'$.

3) let's determine an argument of the complex number $\frac{Z_{A'}-Z_{B'}}{Z_{D'}} = Z_{E'}$

We've $\frac{Z_{A'}-Z_{B'}}{Z_{D'}} = \frac{2e^{i\frac{\pi}{3}}-2}{e^{\frac{\pi i}{6}}} = \frac{2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i-1\right)}{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)} = \frac{2(1+\sqrt{3}i)}{(\sqrt{3}+i)} = \frac{2e^{\frac{\pi i}{3}}}{e^{i\frac{\pi}{6}}} = 2e^{i\frac{\pi}{3}}.e^{-i\frac{\pi}{6}}$

$$= 2e^{i\left(\frac{\pi}{3}-\frac{\pi}{6}\right)} = 2e^{i\frac{\pi}{6}}$$

thus

$$argument\ of\ Z_{E'} = \frac{\pi}{6}$$

Let's prove that the line (OD') is a median of the triangle $OA'B'$

We have $\left|\frac{Z_{D'}-Z_{B'}}{Z_{D'}-Z_{A'}}\right| = \left|\frac{\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-(2)}{\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)-2e^{i\frac{2\pi}{3}}}\right| = \left|\frac{1-2}{1-4}\right| = \frac{1}{3}$

$since\ \left|\frac{Z_{D'}-Z_{B'}}{Z_{D'}-Z_{A'}}\right| = \frac{1}{3}$, we conclude that OD' is the median of the triangle $OA'B'$ ■

Exercise 2 (5pts)

Given the differential equation $y' + 5y = 0 \dots \dots .. (E)$.

4)let's prove that the function f is the solution of (E), iff the function $F = f$ is the solution of .

We have $f'' + 5f' = 0 \Longrightarrow f$ is a solution of (E).

If $F = f'$ is a solution of $(E_1) \Longrightarrow F' + 5F = f'' + 5f' = 0$

Hence $f\ is\ a\ solution\ of\ (E)\ iif\ F = f'$

5)let's resolve the differential equation (E)

We have $y' + 5y = 0 \Longrightarrow \frac{y'}{y} = -5 \Longrightarrow \int \frac{y'}{y}dy = -\int 5dx$ from where

$\ln|y| = -5x + c \Longrightarrow y = e^{-5x+c} = Ae^{-5x}, (A = e^c) \in \mathbb{R}$.

$$y = Ae^{-5x}, A \in \mathbb{R}$$

6)Given that $g(x) = \text{a}\cos(x) + bsin(x), a, b, and\ x \in \mathbb{R}$, let's determine $a\ and\ b$ such that g checks the differential equation $y'' + 5y' = 26\cos(x)$

We've $g'(x) = -\text{a}\sin(x) + bcos\ and\ g''(x) = -\text{a}\cos(x) - bsin(x)$

Substituting $g'(x) and\ g''(x)\ into\ g(x)$ above, we have

$g''(x) + g'(x) = -[a\cos(x) + b sin(x)] + 5[-a\sin(x) + b cos(x)] \equiv 26\cos(x)$ from where we have,

$[-a\cos(x) + 5b cos(x)] + [-b sin(x) - 5a\sin(x)\ ] = 26\cos(x)$

$\Longrightarrow (-a + b)\cos(x) + (-a - b)\sin(x) = 26\cos(x)$ Equating coefficients of $\sin(x)$ *and* $\cos(x)$ *on the RHS to that on the LHS* of the equation, we have

$$\begin{cases} -a + 5b = 26 \dots\dots (i) \\ -5a - b = 0 \dots\dots (ii) \end{cases} \Rightarrow \begin{cases} a = -\frac{b}{5} \dots\dots (ii) \\ -\left(-\frac{b}{5}\right) + 5b = 26 \end{cases} \Rightarrow \begin{cases} a = -1 \\ b = 5 \end{cases}$$

Thus $g(x) = -\cos(x) + 5\sin(x) \equiv a\cos(x) + b sin(x)$

$a = -1$ *and* $b = 5$

7)let's prove that f is a solution of (E') iff $f - g$ is solution of (E).

If f is a solution of $(E') \Longrightarrow f'' + f' = 26\cos(x) \dots\dots(iii)$

And if $(f - g)$ is a solution of (E)

$\Longrightarrow (f - g)'' - (f - g)' = 0 \Longrightarrow (f'' - 5f') - (g'' - f') = 0$

From (iii), we've $26\cos(x) - (g'' - 5g') = 0 \Longrightarrow (g'' - 5g') = 26\cos(x) \dots\dots (iv)$

(iii)*and* $(iv) \Longrightarrow f$ *is a solution of* E' *iif* $f - g$ *is a solution of* E

8)let's determine all the solutions of E' or the general solution of E'

we have the characteristic equation.

$x^2 + 5x = 0 \Longrightarrow x = 0$ *or* $x = -5$

the particular solution is $y_p = -\cos(x) + 5\sin(x)$

The general solution of E' is $f_g = y_p + y_c = -\cos(x) + 5\sin(x) + Ae^{-5x} + B, A, B \in \mathbb{R}$.

Thus $f_g = -\cos(x) + 5\sin(x) + Ae^{-5x} + B\ , A, B \in \mathbb{R}$.

9)let's determine the solution of E', *checking* $f(0) = 0$ *and* $f'(0) = 0$.

We've $f(x) = -\cos(x) + 5\sin(x) + Ae^{-5x} + B$ *and*

$f'(x) = \sin(x) + 5\cos(x) - 5Ae^{-5x}$

$\Longrightarrow f(0) = -\cos(0) + 5\sin(0) + Ae^{0} + B = -1 + A + B = 0 \Longrightarrow A + B = 1$

$\Longrightarrow f'(0) = \sin(0) + 5\cos(0) - A = 0 \Longrightarrow A = 5$ and $A + B = 1 \Longrightarrow B = 1 - 5 = -4$

Thus $f_g = -\cos(x) + 5\sin(x) + 5e^{-5x} - 4$

Exercise 3 (7pts)

Given the function $f(x)$, not nil defined by $f(x) = \frac{2e^x+3}{e^x-1}$ and its curve C_f

1.let's study limit of $f\ at - \infty\ and\ at + \infty$

At $+\infty$, we have

$\lim_{x\to-\infty}[\mathrm{f(x)}] = \lim_{x\to-\infty}\left[\frac{2e^x+3}{e^x-1}\right] = -3, since\ as\ x \to -\infty, e^x \to 0$

At $+\infty$, we have

$\lim_{x\to+\infty}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{x\to+\infty}\left[\frac{2+\frac{3}{e^x}}{1-\frac{1}{e^x}}\right] = 2\ ; since\ as\ x \to +\infty, \frac{3}{e^x} \to 0\ and\ \frac{-1}{e^x} \to 0$

$\lim_{x\to-\infty} f(x) = -3\ and\ \lim_{x\to+\infty} f(x) = 2$

2.let's study the limits of f as x turns to zero by inferior value 0^- and by superior values 0^+

At 0^-, we have

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left[\frac{2e^x+3}{e^x-1}\right]$ let $x = \frac{1}{X}$ $then\ as\ x \to 0^-;\ X \to -\infty\ and\ \frac{1}{X} \to 0$

$\lim_{x\to0^-} f(x) = \lim_{x\to0^-}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{X\to-\infty}\left[\frac{2e^{\frac{1}{X}}+3}{e^{\frac{1}{X}}-1}\right] = -3$

At 0^+, we have $\lim_{x\to0^+} f(x) = \lim_{x\to0^+}\left[\frac{2e^x+3}{e^x-1}\right] = \lim_{X\to+\infty}\left[\frac{2+\frac{3}{e^{\frac{1}{X}}}}{1-\frac{1}{e^{\frac{1}{X}}}}\right] = +\infty$

3)let's deduce the asymptotes of C_f

4)let's deduce the derivative of f and study the variation of f.

$* \forall x \in \mathbb{R};\ f$ is derivable$\Longrightarrow f'(x) = \left[\frac{2e^x+1}{e^x-1}\right]' = \frac{-3e^x}{(e^x-1)^2}$

Thus $\forall x \in \mathbb{R}, f'(x) = \frac{-3e^x}{(e^x-1)^2}$

$*$Variation of f.

$\forall x \in]-\infty; 0[,\ (e^x-1)^2 > 0\ and - 3e^x < 0 \Longrightarrow \frac{-3e^x}{(e^x-1)^2} = f'(x) < 0$ …….(v)

$\forall x \in]0;\ +\infty[, (e^x-1)^2 > 0\ and - 3e^x < 0 \Longrightarrow \frac{-3e^x}{(e^x-1)^2} = f'(x) < 0$ …..(vi)

$Therefor, \forall x \in \mathbb{R} - \{0\}; f'(x) < 0$ and it's strictly decreasing on $\mathbb{R} - \{0\}$

Exercise 4 (4pts)

There are four affirmations let's say which of them is true or false, given that

$f(x) = \ln\left[\frac{2x+1}{x-1}\right]$

1. f *is defined on* $]1; +\infty[$, **true** it's defined on $]-\infty; 0[$ and on $]1; +\infty[$.
2. $f'(x) = -\frac{1}{(x-1)^2}\ln\left(\frac{2x+1}{x-1}\right)$, ***false***, $f'(x) = \frac{2x-3}{(2x+1)(x-1)}$
3. The line $x = 1$ is an asymptote to the curve of f. **false.**
4. The curve of f admet a horizontal asymptote. **True**

JULY 2011

Exercise I (7mks)

A)Let f be a function defined on $]0; +\infty[$ by $f(x) = x - 2 + \frac{1}{2}\ln x$.

1)a-let's calculate the limits of f at 0 and at +

$At\ 0, we\ have, \lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x - 2 + \frac{1}{2}\ln x\right] = -\infty$

At $+\infty$, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 2 + \frac{1}{2}\ln x\right] = +\infty$.

Thus

$\lim_{x\to 0} f(x) = -\infty\ and\ \lim_{x\to+\infty} f(x) = +\infty$

b)let's calculate $f'(x)$ and give the table of variation of f.

$\forall x \in]0; +\infty[, f\ is\ differentiable \Longrightarrow f'(x) = 1 + \frac{1}{2x} = \frac{2x+1}{2x}$

$\forall x \in]0; +\infty[, f'(x) = \frac{2x+1}{2x}$

We know from above that $x > 0 \Longrightarrow \forall x \in]0; +\infty[, \frac{2x+1}{2x} > 0$

Table of variation

$\forall x \in]0; +\infty[\ , f'(x) > 0, it's\ strickly\ increasing\ on\ ]0; +\infty[$

x		0	$\frac{-1}{2}$	$+\infty$
$f'(x)$			$+$	
$f(x)$		$-\infty$ ↗	0 ↗	$+\infty$