Exercise 4 (4pts)

There are four affirmations let's say which of them is true or false, given that

$f(x) = \ln\left[\frac{2x+1}{x-1}\right]$

1. $f\ is\ defined\ on$ $]1;\ +\infty[$, **true** it's defined on $]-\infty; 0[$ and on $]1;\ +\infty[$.
2. $f'(x) = -\frac{1}{(x-1)^2}\ln\left(\frac{2x+1}{x-1}\right)$, ***false***, $f'(x) = \frac{2x-3}{(2x+1)(x-1)}$
3. The line $x = 1$ is an asymptote to the curve of f. **false.**
4. The curve of f admet a horizontal asymptote. **True**

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Exercise I (7mks)

A)Let f be a function defined on $]0;\ +\infty[$ by $f(x) = x - 2 + \frac{1}{2}\ln x$.

1)a-let's calculate the limits of f at 0 and at +

$At\ 0, we\ have,$ $\lim_{x\to 0} f(x) = \lim\limits_{x\to 0}\left[x - 2 + \frac{1}{2}\ln x\right] = -\infty$

At $+\infty$, we have $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 2 + \frac{1}{2}\ln x\right] = +\infty$.

Thus

$$\boxed{\lim_{x\to 0} f(x) = -\infty\ and\ \lim_{x\to+\infty} f(x) = +\infty}$$

b)let's calculate $f'(x)$ and give the table of variation of f.

$\forall x \in]0;\ +\infty[, f\ is\ differentiable \Longrightarrow f'(x) = 1 + \frac{1}{2x} = \frac{2x+1}{2x}$

$$\boxed{\forall x \in]0;\ +\infty[, f'(x) = \frac{2x+1}{2x}}$$

We know from above that $x > 0 \Longrightarrow \forall x \in]0;\ +\infty[, \frac{2x+1}{2x} > 0$

Table of variation

$$\boxed{\forall x \in]0;\ +\infty[\ , f'(x) > 0, it's\ strickly\ increasing\ on\ ]0;\ +\infty[}$$

x		0	$\frac{-1}{2}$	$+\infty$
$f'(x)$			$+$	
$f(x)$		$-\infty$	$\nearrow 0 \nearrow$	$+\infty$

4)a) let's show that the equation $f(x)=0$, has a unique solution denoted by α on $]0;\ +\infty[$

x	1.75	1.74	1.73	1.72
$f(x)$	0.029	0.016	0.004	-0.001

f is defined and continuous and also strictly increasing in the interval $]0;\ +\infty[$

$f(]0;\ +\infty[) =]\lim_{x\to 0} f(x);\ \lim_{x\to+\infty} f(x)[=]-\infty;\ +\infty[$ and $0 \in]-\infty;\ +\infty[$,

Therefore $\boxed{f(x)=0 \ has\ a\ unique\ solution\ \alpha\ on\ ]0;\ +\infty[}$

Let's give the value of $\alpha\ at 10^{-2}\ near$.

We have $f(1)=1-2+\frac{1}{2}\ln 1=-1$ and $f(2)=2-2+\frac{1}{2}\ln 2=0.35$

$f(1).f(2)<0 \Rightarrow \alpha \in]1;2[$,

$\frac{1+2}{2}=1.5$

Also $f(1.5)=1.5-2+\frac{1}{2}\ln 1.5=-0.29$

$f(2).f(1.5)<0, \Rightarrow \alpha \in]1;2[$

$\frac{1.5+2}{2}=1.75 \Rightarrow f(1.75)=1.75-2+\frac{1}{2}\ln 1.75=0.029$

$f(1.5).f(1.75)<0 \Rightarrow \alpha \in]1.5, 1.75[$

$\frac{1.5+1.75}{2}=1.625 \Rightarrow f(1.625)=1.625-2+\frac{1}{2}\ln 1.625=-0.13$

$f(1.75).f(1.625)<0 \Rightarrow \alpha \in]1.625; 1.75[$

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$f(1.72).f(1.73)<0, \Rightarrow \alpha \in]1.72; 1.73[\ since\ [1.73-1.72]=10^{-2}$

B)let's g be a function defined on $[0;\ +\infty[$ by $g(x)=-\frac{7}{8}x^2+x-\frac{1}{4}x^2\ln x, \forall x>0, g(0)=0$

1) a) let's study the continuity and the differentiation of $g\ in\ 0$.

$\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}\left[-\frac{7}{8}x^2+x-\frac{1}{4}x^2\ln x\right]=0=g(0)$ thus

$\boxed{g(0)=\lim_{x\to 0^+} g(x)=0, \text{ it's continuous at } 0}$

$\lim_{x\to 0^+}\left[\frac{g(x)-g\prime(0)}{x-0}\right]=\lim_{x\to 0^+}\left[-\frac{7}{8}x+1-\frac{1}{4}x\ln x\right]=1$

Since $\lim_{x\to 0^+}\left[\frac{g(x)-g'(0)}{x-0}\right]$ ∃!, we conclude that g is differentiable at 0

2)a) Let's calculate $g'(x) and\ verify\ that\ g'(x) = xf\left(\frac{1}{x}\right), \forall x > 0$

From above $g(x)$ is differentiable, thus $g'(x) = \left(-\frac{7}{8}x^2 + x - \frac{1}{4}x^2 \ln x\right)'$

$$= -\frac{7}{4}x + 1 - \frac{1}{2}x(\ln x) - \frac{1}{4}x$$

$$= x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$$

Thus $g'(x) = x\left[-2 + \frac{1}{x} + \frac{1}{2}\ln\left(\frac{1}{x}\right)\right] = xf\left(\frac{1}{x}\right)$, $\forall x > 0$

x	0		k		$+\infty$
$g'(x)$		$+$		$-$	
$g(x)$	0	$\nearrow$	$g(k)$	$\searrow$	$-\infty$

b)Let's deduce the sign of $g'(x)$ and draw the variation table.

$\forall x \in]0; k], g'(x) > 0\ it's\ strctly\ increasing\ on\ ]0; k]$

$\forall x \in [k;\ +\infty[, g'(x) < 0, it's\ strictly\ decreasing\ on\ [k;\ +\infty[$

4)let's give the equations of tangent to the curve of g at the point of $x - coordinate\ 0\ and\ 1$

At the point, $x = 0$, we have

$y_0 = g(0)(x - 0) + g(0) = x$

At the point, $x = 1$, we have

$y_1 = g(1)(x-1) + g(1) = -x + 1 + \frac{1}{8} = -x + \frac{9}{8}$, thus equations are; $\boxed{y_0 = x \; and \; y_1 = -x + \frac{9}{8}}$

$*$ let 's plot $c \; and \; the \; tangents$

Exercise II

a)Let's evaluate the following,s

1) $\int_0^{\sqrt{3}} x^2 \arctan(x)dx$.

$Let \begin{cases} u = \arctan(x) \\ v' = x^2 \end{cases} \Longrightarrow \begin{cases} u' = \frac{1}{1+x^2} \\ v = \frac{1}{3}x^3 \end{cases}$ Using integration by parts we have,

$$\int_0^{\sqrt{3}} x^2 \arctan(x)\,dx = \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{2}} \frac{x^3}{1+x^2}dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]_0^{\sqrt{3}} - \frac{1}{3}\int_0^{\sqrt{3}} \left[x - \frac{x}{1+x^2}\right]dx$$

$$= \left[\frac{1}{3}x^3\arctan(x)\right]^{\sqrt{3}} - \frac{1}{3}\left[\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2)\right]_0^{\sqrt{3}}$$

$$= \frac{1}{3}\left(\sqrt{3}\right)^3 \arctan\left(\sqrt{3}\right) + \frac{3}{2} - \ln 2$$

$= \frac{\sqrt{3}\pi}{3} + \frac{3}{2} - \ln 2$, thus we have $\boxed{\int_0^{\sqrt{3}} x^2 \arctan(\quad \frac{\pi\sqrt{3}}{3} + \frac{3}{2} - \ln 2}$

2) $\int_0^1 \frac{1}{(1+x^2)^2}dx$

Let, $x = tg(t)$, for $x = 0, t = 0 \; and \; for \; x = 1, t = \frac{\pi}{4}$

$$dx = \frac{1}{cos^2(t)}dt \Longrightarrow \int_0^1 \frac{1}{(1+x^2)^2}dx = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{cos^{2(t)}}}{(1+(tg)^2)^2}dt$$

$$= \int_0^{\frac{\pi}{4}} cos^2(t)dt = \int_0^{\frac{\pi}{4}} \frac{1+cos2t}{2}dt = \frac{1}{2}\left[t + \frac{1}{2}sint\right]_0^{\frac{\pi}{4}} = \frac{\pi+2}{8}$$

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Thus

$$\boxed{\int_0^1 \frac{1}{(1+x^2)^2}dx = \frac{\pi+2}{8}}$$

Exercise III

b)Let's solve the following differential equations.

$y' + y = 2\cos(x) + (x+1)e^{-x}$……………$E$

∗ let's determine the characteristic equation of y.

We have, $y' + y = 0 \Longrightarrow \frac{y'}{y} = -1 \Longrightarrow \int \frac{1}{y} dy = \int(-1)dx \Longrightarrow \ln y_c = -x + k \Longrightarrow y_c = e^t e^{-x} = ke^{-x}\ k \in \mathbb{R}$.

∗ let's determine the particular solution of

Let $y_c = A(x)e^{-x} \Longrightarrow y'_c = A'(x)e^{-x} - A(x)e^{-x}$. Substituting y_c *and* y'_c into E, we have

$A'(x)e^{-x} - A(x)e^{-x} + A(x)e^{-x} = 2\cos(x) + (1+x)e^{-x}$

$\Longrightarrow A'(x)e^{-x} = 2\cos(x) + (x+1)e^{-x} \ldots.(E)$

Multiplying all through by e^x we have $A'(x) = 2e^x\cos(x) + (1+x)$. Integrating one has

$\int A'(x)dx = \int[2e^x\cos(x) + (1+x)]dx \Longrightarrow A(x) = 2\int e^x\cos(x)\,dx + x + \frac{x^2}{2}$ …………E'

From E', we have $\int e^x\cos(x)\,dx = e^x\cos(x) + \int e^x\sin(x)\,dx$

$$= e^x[\cos(x) + \sin(x)] - \int e^x\cos(x)\,dx + k$$

$$\Longrightarrow 2\int e^x\cos(x)\,dx = e^x[\cos(x) + \sin(x)] + k \Longrightarrow \int e^x\cos(x)\,dx = \frac{e^x}{2}[\cos(x) + \sin(x)] + c \ldots E''$$

Substituting E'' into E', we have

$A(x) = e^x[\cos(x) + \sin(x)] + x + \frac{x^2}{2} + c, c \in \mathbb{R}$. Thus general

Solution is $y = e^x[\cos(x) + \sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, k\ and\ c \in \mathbb{R}$.

$$\boxed{y = e^x[\cos(x) + \sin(x)] + \frac{x^2}{2} + x + ke^{-x} + c, \qquad \forall k, c \in \mathbb{R}.}$$