SEPTEMBER 2012
Exercise I (1.5+1+1.5=4)

Given that $\int_0^{\frac{\pi}{2}} cos^n\,(x)dx$

1-Let's calculate I_0 , $I_1 and\ I_2$.

We have $\int_0^{\frac{\pi}{2}} cos^n\,(x)dx$, so when $n = 0,1,2, we$ have respectively

$I_0 = \int_0^{\frac{\pi}{2}} cos^0(x)dx = \int_0^{\frac{\pi}{2}} 1dx = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

$$I_1 = \int_0^{\frac{\pi}{2}} cos^1(x)dx = [\sin(x)]_0^{\frac{\pi}{2}} = sin\frac{\pi}{2} - sin0$$

$$I_2 = \int_0^{\frac{\pi}{2}} cos^2\,(x)dx = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos(2x))\,dx$$

$I_2 = \frac{1}{2}\left[x + \frac{1}{2}sin2x\right]_0^{\frac{\pi}{2}} = \frac{1}{2}\left[\frac{\pi}{2} + \frac{1}{2}sin2\left(\frac{\pi}{2}\right) - 0 - sin0\right] = \frac{\pi}{4}$ Therefore we have $\boxed{I_0 = \frac{\pi}{2},\ \ I_1 = 1\ and\ I_2 = \frac{\pi}{4}}$

b) Let's show that $\forall n \geq 2, one\ has\ nI_n = (n-1)I_{n-2}$

We suppose $U(x) = cos^{n-1}(x)\ and\ V'(x) = \cos(x). Also\ cos^n(x) = cos^{n-1}(x)\cos(x)$

$\begin{cases} U'(x) = -(n-1)cos^{n-2}(x)\sin(x) \\ V(x) = \sin(x) \end{cases}$

Using integration by parts we have $I_n = \mathrm{V(x).U(x)} = \int_0^{\frac{\pi}{2}} \mathrm{U'(x)\,V(x)dx} \ldots\ldots$ (i) , substituting $U(x), U'(x) and\ V'(x), V(x)\ \ into\ (i)$ gives

$I_n = [\mathrm{\sin(x)cos^{n-1}(x)}]_0^{\frac{\pi}{2}} + (\mathrm{n}-1)\int_0^{\frac{\pi}{2}} \mathrm{\sin(x)\,cos^{n-2}(x)\,\sin(x)\,dx}$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} sin^2\,(x)cos^{n-2}(x)dx = (n-1)\int_0^{\frac{\pi}{2}}(1 - cos^2(x))\,cos^{n-2}(x)dx$

$= (n-1)\int_0^{\frac{\pi}{2}} cos^{n-2}(x)dx - (n-1)\int_0^{\frac{\pi}{2}} cos^n\,(x)dx = (n-1)I_{n-2} - (n-1)I_n$

$\Leftrightarrow I_n = (n-1)I_{n-2} - (n-1)I_n \Leftrightarrow I_n + (n-1)I_n = nI_n = (n-1)I_{n-2}$

Thus $\boxed{nI_n = (n-1)I_{n-2}}$

c) Let's deduce the value of $I_n\ \forall n \geq 1$

From (b) we have $nI_n = (n-1)I_{n-2} \Leftrightarrow \frac{I_n}{I_{n-2}} = \frac{n-1}{n} = 1 - \frac{1}{n} \Leftrightarrow n = \frac{I_{n-2}-I_n}{I_{n-2}}$.

Thus we have $\boxed{\frac{I_{n-2}-I_n}{I_{n-2}} = n\ \ \ \forall n \geq 1}$

Exercise II (4mks)

1)Let's study the continuity of the following function

$$f(x) = \begin{cases} x^2\ if\ x \leq 0 \\ x\ \ if\ 0 < x < 2 \\ 4 - x\ \ if\ x \geq 2 \end{cases} \longrightarrow \begin{cases} g(x) if\ x \leq 0 \\ h(x)\ if\ 0 < x < 2 \\ i(x)\ \ if\ x \geq 2 \end{cases}$$

∗We have $\lim_{x\to 0^-} x^2 = 0 = g(0)$ hence is continuous $\forall x \leq 0$

∗Let $x = 1 \in]0;2[$ then we have $\lim_{x\to 1} x = 1 = h(1)$ thus $h(x) is\ continuous\ \forall x \in]0;2[$

∗Lastly, we have $\lim_{x\to 2^+} 4 - x = 2 = i(2)$, hence $i(x)\ is\ continuous\ \forall x \geq 2$

$\boxed{\text{Since} f(x) is\ \ \text{continuous}\forall x \leq 2;\ \forall x \in]0;2[\ and\ \forall \geq 2\ we\ conclude\ it's\ continuous\ \forall x \in \mathbb{R}}$

2) Let's show $g(x)$ is continuous and derivable and it's derivative $g'(x)$is continuous

We've $g(x) = \begin{cases} x^2 \ \ if\ x \leq 0 \\ 0 \ \ if\ x = 0 \\ \cos(x) - 1\ if x > 0 \end{cases} \Leftrightarrow \begin{cases} h(x) if\ x < 0 \\ g(x)\ if\ x = 0 \\ i(x)\ if\ x > 0 \end{cases}$

$*$ We've $\lim_{x\to0^-} x^2 = 0 = \lim_{x\to0^+}(cos(x) - 1)\ hence\ g(x) is\ cotinuous \ldots\ldots\ldots\ldots (i)$

$g'(x) = \begin{cases} -\frac{1}{x^2}e^{\frac{1}{x}}\ if\ x < 0 \\ 0\ \ if\ x = 0 \\ -\sin(x)\ if\ x > 0 \end{cases}$

$*$We've $\lim_{x\to0^-}\frac{h(x)-h'(0)}{x-0} = \lim_{x\to0^-}\frac{e^{\frac{1}{x}}+\frac{e^{\frac{1}{x}}}{x^2}}{x} = \lim_{x\to0^-}\frac{e^{\frac{1}{x}}(1+\frac{1}{x^2})}{x}$

Let $X = \frac{1}{x} \Leftrightarrow As\ x \longrightarrow 0^-, X \longrightarrow -\infty$ Substituting X in the above we've

$\lim_{x\to0^-}\frac{e^{\frac{1}{x}}(1+\frac{1}{x^2})}{1/\frac{1}{x}} = \lim_{X\to-\infty}\frac{e^X(1+X^2)}{\frac{1}{X}} = \lim_{X\to-\infty} Xe^X(1 + X^2) = 0$

Also from $g'(x)$ we've $\lim_{x\to0^+}\sin(x) = 0$

Therefore $g(x)$ differentiable $\ldots\ldots\ldots (ii)$

$*\ Determination\ of\ continuity\ of\ g'(x)$

$\boxed{from\ (i) limits\ of\ g(x)\ exist\ and\ from\ (ii)\ g(x)\ is\ respectively\ continuous\ and\ derivable\ and\ g'(\ x)\ is\ continuous\ from\ (iii)}$

$We\ \ have\ \ lim_{x\to0^-}(-\frac{1}{x^2}e^{\frac{1}{x}}) = 0 = lim_{x\to0^+} sin(x)\ \ from\ g'(x)\ above\ \ldots\ldots\ldots\ldots (iii)$

Exercise III (3mks)

Given a discrete random variable X with the probability distribution with $E(X) = 2$, with distribution table below.

$(X = x)$	0	1	2	3	4
$P(X = x)$	0.1	p	0.25	q	0.05

a) Let's find the values of $p\ and\ q$

We know that for a discrete random variable, $X, \sum_{i=0}^{4} x_i P(X = x_i) = 1$. Hence we have

$\sum_{i=0}^{4} x_i\ P(X = x_i) = 0x0.0.1 + 1xp + 2x0.25 + 3xq + 4x0.05 = E(X) = 2$

$\Leftrightarrow 1xp + 2x0.25 + 3xq + 4x0.05 = 2 \Leftrightarrow p + 3q = 1.3 \ldots\ldots (i)$

Also $\sum_{i=0}^{4} P(X = x_i) = 0.1 + +p + 0.25 + q + 0.05 = 1 \Leftrightarrow p + q = 0.6 \ldots\ldots\ldots (ii)$

$\begin{cases} p + 3q = 1.3 \ldots\ldots\ldots (i) \\ p + q = 0.6 \ldots\ldots\ldots\ldots (ii) \end{cases} \Leftrightarrow \begin{cases} p = 0.35 \\ q = 0.25 \end{cases}$ By solving $(i) and\ (ii)$

Therefore we have

$\boxed{p = 0.25\ and\ q = 0.35}$

b) Let's find $Var(X)$.We know that, $Var(X) = E(X^2) - E^2(X) = E(X^2) - \mu^2$, where $E(X^2) = \sum_0^4 x^2\ P(X = x) \Leftrightarrow E(X^2) = 0^2x0.1 + 1^2x0.25 + 2^2x0.25 + 3^2x0.35 + 4^2x0.05 = 5.00$

$Var(X) = E(X^2) - E^2(X) = (5.00) - (2.02)^2 = 5.00 - 4.81 = 0.19.$

Therefore $\boxed{Var(X) = 0.19}$ we have

c) Let's calculate the average $E(Y)$ and variance $Var(Y)$ of $Y = 5X + 4$

$*$Average of Y

$E(Y) = E(5X+4) = 5E(X) + 4 = 5x2.02 + 4 = 10.10 + 4 = 14.10 \ldots\ldots(1)$

$*$ Variance of Y

$Var(Y) = Var(5X+4) = Var(5X) + Var(4)$

$= 5^2 Var(X) + 0 = 25x0.19 = 4.75 \ldots.(2)$

Therefore $\boxed{(1) and\ (2) \Longrightarrow E(Y) = 14.10\ and\ Var(Y) = 4.75}$

Exercise III (9mks).
Part 1 (6mks).

Let f be a function defined on $\mathbb{R}$ by $f(x) = \frac{3(x-1)^3}{3x^2+1}$ and let C be its curve

1) Let's that there exists a single triplet $(a; b; c)$ that one will determine such as for real x

$f(x) = ax + b + \frac{cx}{3x^2+1}$

We have $f(x) = \frac{3(x-1)^3}{3x^2+1} = \frac{3x^3-9x^2+9x-3}{3x^2+1}$ by expansion and using long division method one has

$\frac{3x^3-9x^2+9x-3}{3x^2+1} = x - 3 + \frac{8x}{3x^2+1} \equiv ax + b + \frac{cx}{3x^2+1}$..........(I')

Therefore we have

$$\boxed{From\ (I')\ that\ f(x) = x - 3 + \frac{8x}{1+3x^2} \Longrightarrow a = 1, b = -3\ and\ c = 8}$$

2) Let's determine the limits of $f\ in\ \pm\infty$

At $+\infty$, $\text{Lim}_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[x - 3 + \frac{8x}{3x^2+1}\right]$

$= \lim_{x\to+\infty} x\left[1 - \frac{3}{x} + \frac{8}{3x+\frac{1}{x}}\right] = +\infty\ since\ \frac{1}{x} \to 0. as\ x \to +\infty$

At $-\infty$, $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(x - 3 + \frac{8x}{3x^2+1}\right) = \lim_{x\to-\infty} x\left[1 - \frac{3}{x} + \frac{8}{3x+\frac{1}{x}}\right] = -\infty$

Therefore

$$\boxed{\lim_{x\to+\infty} f(x) = +\infty\ and\ \lim_{x\to-\infty} f(x) = -\infty}$$

3) Let's show that f is differentiable and calculate its derivative

$*$ f is made up of three parts $x - 3, 8x\ and\ 3x^2 + 1$ all differentiable on $\mathbb{R}\ or\ on\ ]-\infty;\ +\infty[$ therefore $f\ is\ itself\ differentiable\ on\ \mathbb{R}\ or\ on\ ]-\infty;\ +\infty[\ \ldots\ldots\ldots\ldots(a)$

Hence $\forall x \in\ ]-\infty;\ +\infty[\ or\ \mathbb{R}$, we have $f'(x) = \left[x - 3 + \frac{8}{3x^2+1}\right]' = 1 + \frac{24x^2-48x+8}{(3x^2+1)^2}\ \ldots\ldots..(b)$.

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$$\boxed{From\ (a), f(x)\ is\ differentiable\ on\ \mathbb{R}\ and\ from\ (b), f'(x) = 1 + \frac{24x^2 - 48x + 8}{(3x^2+1)^2}}$$

4) Let's draw the table of variation of f.

$\forall x\epsilon\ ]-\infty; 0]\ \ f'(x) > 0, hence\ positive$ and $\forall x\epsilon[0;\ +\infty[\ f'(x) > 0\ hence\ increasing.$

There no turning points since x has no real value(s). Intercepts: when $x0\ subsituting\ x = 0\ into\ y = f(x)\ gives\ y = \frac{3(0-1)^3}{3(0)^2+1} = -3$. hence $(0;\ -3)$.

When $f(x) = 0\ we've\ 0 = \frac{3(x-1)^3}{3x^2+1} \Longleftrightarrow x = 1, hence\ (1; 0)$

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$		$+$	
$f(x)$		↗	0	↗	$+\infty$

+

	+ +

5) Let's show that the curve (C) has the line $(D): y = x - 3$, as oblique asymptote $\lim_{x\to\pm\infty}[f(x) - y] = \lim_{x\to\pm\infty}\left[x - 3 + \frac{8x}{1+3x^2} - (x-3)\right] = \lim_{x\to\pm\infty}\left[\frac{8x}{3x^2+1}\right] = 0 \ldots.(k)$

Therefore

$\boxed{The\ line\ y = x - 3\ is\ an\ oblique\ asymptote\ to\ (C)\ \ from\ (k)}$ ■

6) Let's study the relative positions

We have $f(x) - y = \frac{8x}{3x^2+1}$ we know that $\forall x \in \mathbb{R}, x^2 > 0\ and\ 3x^2 + 1 > 0$. therefore the sign of $f(x) - y\ depends\ on\ that\ of\ 8x,\ \ and$ we have that

$$\forall x \in]-\infty; 0], 8x < 0\ and\ \frac{8x}{3x^2+1} < 0 \Longrightarrow f(x) - y < 0, thus\ negative, \forall x \in]-\infty; 0] \ldots\ldots\ldots..(1)$$

$$\forall x \in [0;\ +\infty[\ 8x > 0\ and\ \frac{8x}{3x^2+1} > 0, hence\ f(x) - y, thus\ positive \ldots\ldots.\ \ (2)$$

$\boxed{From\ (1)\ f(x) < y\ hence f(x) is\ below\ y\ \forall x \in]-\infty; 0]\ and\ from\ (2)\ \ f(x) > y,\ hence\ f(x) is\ above\ y, \forall x \in [0;\ +\infty[}$

7) Let's give the equation of tangent (T) to (C) at the point of x −coordinate 0. And trace $(T), (\ D) and\ (C)$.

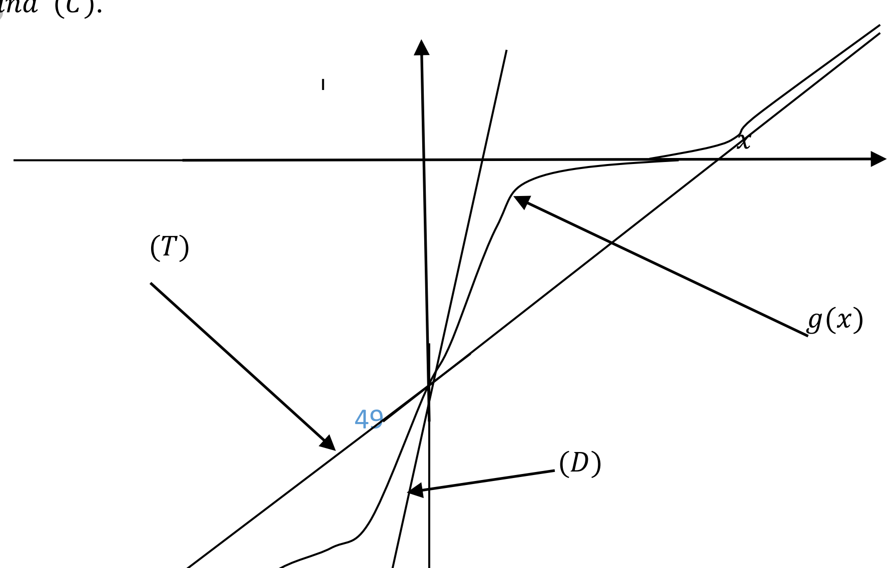
An equation of tangent to a curve at x_0 is given by $y - f(x_0) = f'(x_0)[x - x_0];\ x_0 = 0$

Substituting for $x_0 = 0\ in\ the\ above\ we've$

$$y - \left[\frac{3(x_0-1)^3}{3{x_0}^2+1}\right] = \left[1 + \frac{24x_0 - 48x_0 + 8}{(3x_0^2+1)^2}\right](x - x_0) = y + 3 = 9x;\ \ x_0 = 0 \Leftrightarrow y = 9x - 3$$

$\boxed{equation\ of\ tangent\ is\ y = 9x - 3}$

Let's trace $(D), (T)\ and\ \ (C)$.



8) Let's show that the curve(C) has the centre of symmetry.
9) Let's show that the equation $f(x) = 1$ has a single solution in $\mathbb{R}$ denoted as α.
10) Let's give the approximate value of $\alpha\ to\ 10^{-2}$ nearby excess.

Part II (3mks)

Given the function f defined by on $\mathbb{R}$ by $g(x) = \frac{3(\sin(x)-1)^3}{3sin^2(x)+1}$

1) Let's show that g is differentiable on $\mathbb{R}$ and calculate $g'(x)$

$*\ g$ consists of functions $3(\sin(x) - 1)^3\ and\ sin^2(x) + 1$ which are all differentiable on $\mathbb{R}$ thus g is differentiable on $\mathbb{R}$ …………..(i)

$*\ g'(x) = \left(\frac{(\sin(x)-1)^3}{\sin^2(x)+1}\right)' = \frac{2\cos(x)[sin^2(x)\cos(x)-3sin^2(x)-15\sin(x)+9\cos(x)]}{(sin^2(x)+1)^2}$ $\forall x \in \mathbb{R}$. Using quotient rule.

Therefore $g'(x) = [9cosx(sinx - 1)^2(\sin^2 x + 2sinx + 1)]/(3\sin^2 x + 1)^2$

$= \frac{9cosx(sinx-1)^2(sinx+1)^2}{(3sinx+1)^2}$.

Therefore

$$\boxed{g'(x) = \frac{9cosx(sinx - 1)^2(sinx + 1)^2}{(3sinx + 1)^2}}$$

2. Let's draw the table of variation of

At turning points $g'(x) = 0 \Longrightarrow \frac{9cosx(sinx-1)^2(sinx+1)^2}{(3sinx+1)^2} = 0\ and\ get$

$cosx = 0 \Longrightarrow x = (2n + 1)\pi, \sin x = 1, -1. We\ have\ x \in \left\{\frac{-\pi}{2}; \frac{\pi}{2}\right\}\ in\ the\ range\ [-\pi, \pi]$

$g(-\pi) = -3\ and\ g(\pi) = -3$ and $g(0) = -3$

x	$-\pi$	$\frac{-\pi}{2}$		$\frac{\pi}{2}$	π
$g'(x)$	$-$	0	$+$	0	$-$
$g(x)$	↘	-6	↗	0	↘

3) Let's draw or plot a new drawing the representative curve of g.