JULY 2010

Exercise 1 (4mks)

1$*$ Let's put $f(t)$ in the form $f(t) = e^{kt}$

We have $f(t) = 2^{-t} = 2^{-t.\ln 2} = e^{(-\ln 2)t} = e^{kt}, k = -\ln 2;$

Therefore

$$f(t) = e^{kt} = e^{(-\ln 2)t}, k \in \mathbb{R}$$

$**$ Let's evaluate $\int f(t)dt$

We have $\int f(t)dt = \int e^{(-\ln 2)t}\, dt = -\frac{1}{\ln 2} e^{(-\ln 2)t} + c; c\mathbb{R},$

Thus we have

$$\int f(t)dt = -\frac{1}{\ln 2} 2^{-2} + c; c \in \mathbb{R}$$

2) Let's that the sequence $(U_n)_{n\geq 1}$ defined by $U_n = \int_{-n}^{n} 2^{-2} dt$ is geometrical with common ratio $\frac{1}{2}$

If we have $\frac{U_{n+1}}{U_n} = \frac{1}{2}$, then U_n is a geometrical sequence ;

We have $U_n = \frac{1}{\ln 2}(e^{-n} - e^n) \iff U_{n+1} = \frac{1}{\ln 2}(2^{-(n+1)} - 2^{(n+1)})$ therefore we have

$\frac{U_{n+1}}{U_n} = \frac{(2^{-(n+1)} - 2^{(n+1)})}{2^{-n} - 2^n} = \frac{\frac{1}{2}(2^{-n} - 4.2^n)}{2^{-n} - 2^n} \neq \frac{1}{2}$, thus we have

$$since\ \frac{U_{n+1}}{U_n} \neq \frac{1}{2};\ (U_n)_{n\geq 1} \text{ is not a geometrical sequence}$$

3) Given that, $S_n = \sum_{i=1}^{n} U_i$, Let's expressS_n in terms of n.

We have $S_n = U_1 + U_2 + \cdots + U_n$ Since the sequence$(U_n)_{n\geq 1}$ is not a geometrical sequence we cannot find S_n in term of n.

Exercise II (3mks)

1-Let's solve the differential equation, $\frac{d^2y}{dx^2} - 16y = 0 \dots \dots (E), given\ that\ y = 1\ \frac{dy}{dx} = 0\ when\ \ x = 0.$

Let the characteristic equation of $be\ r^2 - 16 = 0 \iff (r+4)(r-4) = 0 \Rightarrow r = \pm 4$. Therefore the particular solution of (E) is the set of function of the form:

$y \longmapsto Ae^{4x} + Be^{-4x}; A, B \in \mathbb{R} \dots\dots\dots (E')$

We have $\frac{dy}{dx} = 4Ae^{4x} - 4Be^{-4x} \dots\dots\dots\dots (E'')$

Substituting $y = 1, \frac{dy}{dx} = 0\ when\ x = 0$ into $(E') and\ (E'')$ give us

$$\begin{cases} A + B = 1 \dots\dots (i) \\ 4A - 4B = 0 \dots\dots (ii) \end{cases}$$

From (ii) $A = B$ and substituting into (i) gives us $A + A = 1$

$\iff A = \frac{1}{2} = B$ Therefore

$$y(x) = \frac{1}{2}(e^{4x} + e^{-4x})$$

2) Let's prove that

$$V = \frac{g}{k} + Ae^{-kt}, where\ k\ is\ a\ constant\ \ and\ given\ that\ \frac{dv}{dt} + kv = 0 \dots\ \ \dots (E)$$

Let homogenous the equation of (E) $be\ \frac{dv}{dt} + kv = 0 \iff \frac{dv}{dt} = -kv$ and collecting like terms we have

$\frac{dv}{v} = -kdt \iff \int \frac{1}{v} dv = \int -kdt$

$\iff \ln v = -kt + c, c\epsilon\mathbb{R}\ \ i.e\ v_h = e^{-kt+c} = e^{-ke}e^c = Ae^{-kt};\ A = e^c = constant$

The particular solution of (E) is obtain when $\frac{dv}{dt} = 0 \Longrightarrow kv_p = g; \Rightarrow v_p = \frac{g}{k}$

The general solution of (E) is $(t) = v_h + v_p = Ae^{-kt} + \frac{g}{k}$; $A, k\epsilon\mathbb{R}$ … … (E''').

Therefore we have

$$v(t) = Ae^{-kt} + \frac{g}{k}, A, k\epsilon\mathbb{R}.$$

3) Given that $v = 0\ when\ t = 0; let's\ find\ t\ when\ v = \frac{g}{2k}$

$v = 0\ when\quad t = 0; 0 = \frac{g}{k} + A \Longrightarrow A = -\frac{g}{k}$ Substituting $A\ and\ v = \frac{g}{2k}$ into (E''') gives $\frac{g}{2k} = \frac{g}{k} - \frac{g}{k}e^{-kt}\ from\ where\ e^{-kt} = \frac{1}{2} \Longrightarrow t = \frac{1}{k}\ln 2$. Therefore

$$t = \frac{1}{k}\ln 2$$

Exercise 3 (4mks)

1)Let's calculate the following quantities

a) $\int \frac{2x}{x^4+1}dx \ldots..(i)$, Let $X = x^2 \Leftrightarrow dX = 2xdx$ subsitutin $X\ and\ dX\ in\ (i), we\ have,$

$$\int \frac{2x}{x^4+1}dx = \int \frac{1}{X^2+1}dX = \arctan(X) + c; c\epsilon\mathbb{R}, but\ X = x^2$$

$$\Leftrightarrow \arctan(X) + c = \tan^{-1}(x^2) + c;\ c\epsilon\mathbb{R}.$$

Therefore, we have

$$\int \frac{2x}{x^4+1}dx = \tan^{-1}(x^2) + c, c\epsilon\mathbb{R}.$$

b) $B = \int_0^1 \frac{t}{1+t^4}dt$

Let $t^2 = X \Leftrightarrow dX = 2tdt, i.e.\quad tdt = \frac{dX}{2}$, Hence B becomes

$$B = \int_0^1 \frac{t}{1+t^4}dt = \frac{1}{2}\int_0^1 \frac{dX}{1+X^2} = \frac{1}{2}(\tan^{-1}X)_0^1 = \frac{1}{2}(\tan^{-1}(x^2))_0^1$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0) = \frac{\pi}{8}$$

Therefore

$$\int_0^1 \frac{t}{1+t^4}dt = \frac{\pi}{8}$$

c) $\int \left[\frac{1}{sin^2(x)} + \frac{1}{cos^2(x)}\right]dx.$

We have $\int \left[\frac{1}{sin^2(x)} + \frac{1}{cos^2(x)}\right]dx = \int \frac{1}{sin^2(x)}dx + \int \frac{1}{cos^2(x)}dx$

$$= \int cosec^2(x)d\,x + \int sec^2(x)dx$$

$= \tan^{-1}(x) + \tan(x) + k,\quad k\epsilon\mathbb{R}.$ Therefore

$$\int \left[\frac{1}{sin^2(x)} + \frac{1}{cos^2(x)}\right]dx = \tan^{-1}(x) + \tan(x) + k;\quad k \in \mathbb{R}$$

d) $D = \int_0^1 \frac{e^x-1}{e^x+1}dx$

We have$D = \int_0^1 \frac{e^{\frac{x}{2}}.e^{\frac{x}{2}} - e^{\frac{x}{2}}.e^{-\frac{x}{2}}}{e^{\frac{x}{2}}e^{\frac{x}{2}} + e^{\frac{x}{2}}e^{\frac{-x}{2}}}dx = \int_0^1 \frac{e^{\frac{x}{2}} - e^{\frac{-x}{2}}}{e^{\frac{x}{2}} + e^{\frac{-x}{2}}}dx = 2\left[\ln\left(e^{\frac{x}{2}} + e^{\frac{-x}{2}}\right)\right]_0^1$

$= 2\left[\ln(e+1) - \frac{1}{2} - \ln 2\right]$ $D\ is\ of\ form\ \int \frac{f'(x)}{f(x)}dx = \ln f(x)$. Thus

$$D = \int_0^1 \frac{e^x-1}{e^x+1}dx = 2\left[\ln\left(\frac{e+1}{2}\right) - \frac{1}{2}\right]$$

Exercise IV

A transformation T of three dimensional space is defined by:

$r' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\quad M\begin{pmatrix} 7 & 5 & 6 \\ 4 & 3 & 3 \\ 10 & 7 & K \end{pmatrix}$ Where $k\ is\ a\ constant.$

A)Let's find the value of k for which there is no inverse transformation matrix.

There will be no transformation matrix if $detM = 0$

$\Leftrightarrow detM = \begin{vmatrix} 7 & 5 & 6 \\ 4 & 3 & 3 \\ 10 & 7 & k \end{vmatrix} = 7(3k-21) - 5(4k-30) + 6(28-30) =$ 0, *from where* $k = 9$

$k = 9$

b)If $k = 9$, let's show that all the points $(x; y; z)$ are transformed into points of the plan $2x' - y' - z' = 0$

When$k = 9$, M^{-1} *or* M −inverse is undefined and $r' = M.r$

$\Leftrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 7 & 5 & 6 \\ 4 & 3 & 3 \\ 10 & 7 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or $r' = T(x) = \begin{cases} x' = 7x + 5y + 6z \ldots\ldots (i) \\ y' = 4x + 3y + 3z \ldots\ldots (ii) \\ z' = 10x + 7y + 9z \ldots\ldots (iii) \end{cases}$ S

$2(i) - (ii) - (iii) \Leftrightarrow 2x' - y' - z' = 2(7x + 5y + 6z) - (4x - 3y - 3z) - (10x + 7y + 9z)$

$\Leftrightarrow (14x - 14x) + (10y - 10y) + (12z - 12z) = 0,$ *Hence*

$2x' - y' - z' = 0$ *when* $k = 9$ *is an equation which transform the points* (x, y, z) *to* (x', y', z')

c) Let's find the value of M^{-1} given that $k = 8$

$$detM = \begin{vmatrix} 7 & 5 & 6 \\ 4 & 3 & 3 \\ 10 & 7 & 8 \end{vmatrix} = 7(24-21) - 5(32-30) + 6(28-30) = -1 \neq 0$$

$$c = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 3 & 2 & -2 \\ -2 & -4 & -1 \\ -3 & -3 & 1 \end{pmatrix}$$

And $(-1)^{i+1}.c^T = \begin{pmatrix} 3 & 2 & -2 \\ -2 & -4 & -1 \\ -3 & -3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -2 \\ 2 & -4 & 1 \\ -3 & 3 & 1 \end{pmatrix}$

$adjM = (-1)^{i+1}.c^T = \begin{pmatrix} 3 & 2 & -3 \\ -2 & 4 & 3 \\ -2 & 1 & 1 \end{pmatrix}$, and

$M^{-1} = \frac{adjM}{detM} = -\begin{pmatrix} 3 & 2 & -3 \\ -2 & 4 & 3 \\ -2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 3 \\ 2 & -4 & -3 \\ 2 & -1 & -1 \end{pmatrix}$

Then

$M^{-1} = \begin{pmatrix} -3 & -2 & 3 \\ 2 & -4 & -3 \\ 2 & -1 & -1 \end{pmatrix}$

d) Let's the point which is mapped into (6; 2; 9).

From ©, we have M^{-1} *and* $M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 9 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = M^{-1} \begin{pmatrix} 6 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 3 \\ 2 & -4 & -3 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ 14 \\ 1 \end{pmatrix}$

Thus

The point is $(x; y; z) = (5, 14, 1)$

Exercise V (4mks)

The distribution table of discrete random variable is given below given that $E(X) = 2.02$

$X = x$	0	1	2	3	4
$P(X = x)$	0.12	p	0.4	q	0.08

a)Let's calculate the value of p *and* q

We know that $E(X) = \sum_{i=0}^{4} x_i P_i \quad , P_i \epsilon [0; 1]$

$= 0x0.12$ +1xp+2x0.4+3xq+4x0.4

$= p + 0.8 + 3q = 2.02 \Longrightarrow p + 3q = 0.9 \ldots\ldots (i)$

Also we know that $\sum_{i=1}^{n} P_i = 1 \Leftrightarrow 0.12 + p + 0.4 + q + 0.08 = 1 \Longrightarrow p + q = 0.4 \ldots\ldots (ii)$

$\Rightarrow \begin{cases} p+3q = 0.9 \ldots\ldots (i) \\ p+q = 0.4 \ldots\ldots (ii) \end{cases}$

$(i)-(ii) \Leftrightarrow 2q = 0.5 \Rightarrow q = 0.25. from\ (ii), p = 0.4 - 0.25 = 0.15.$ Therefore we've $p = 0.15\ and\ q = 0.25$

b) Let's calculate $Var(X)$

P_i	0.12	0.15	0.4	0.25	0.08
X_i	0	1	2	3	4
$X_i - E(X)$	-2.02	-1.02	-0.02	0.98	1.98
$[X_i - E(X)]^2$	4. 0804	1.04404	0.0004	0.9604	3.9204
$[X_i - E(X)]^2 P_i$	0.489648	0.15606	0.00016	0.2401	0.313632

We know that $Var(X) = \sum_{i=1}^{n} [X_i - E(X)]^2\, P_i$

$= 0.489648 + 0.15606 + 0.00016 + 0.2401 + 0.313632 = 1.1996$ $therefore\ Var(X) = 1.1996$

c) Let's calculate the mean and variance of $Y = 3X - 2$

We know that $E(aX+b) = aE(X)+b, where\ a\ and\ b\ are\ constants$

$E(Y) = E(3X-2) = E\left[3\left(X - \frac{2}{3}\right)\right]$

$= 3E\left(X - \frac{2}{3}\right) = 3E(X) - 2$, but $E(X) = 2.02, then\ we've$

$E(Y) = 3x2.02 - 2 = 4.06$

Also we have $Var(aX+b) = a^2 Var(X), similarly$

$Var(Y) = Var(3X-2) = 3^2 Var(X) = 9x1.1996 = 10.7964$

Therefore $E(Y) = 4.06\ and\ Var(Y) = 10.7964$

2) Let's find the probability that a batch is rejected.

The probability that a certain machine is defective

$q = 0.05\ and\ the\ probability\ that\ it's\ not\ defective\ is\ p = 1 - q = 1 - 0.05 = 0.95$

Using Bernouilli law we have $\sum_{k=1}^{n=10} P(X=x) = 1$ and $P(X=K) = C_k^{n=10} p^k q^{n-k}$ $P(X=1) + P(X=2) + \sum_{K=3}^{10} P(X=x) = 1$

$\Leftrightarrow \sum_{k=3}^{10} P(X=x) = 1 - P(X=1) - P(X=2)$

$P(X=1) = C_1^{10}(0.95)(0.05)^9 = 10x0.95(0.05)^9 = 1.5854687x10^{-11}$

Similarly $P(X=2) = C_{10}^{2}(0.95)^2(0.05)^{(10-2=8)} = 1.5864257x10^{-9}$

$\Rightarrow \sum_{k=3}^{n=10} P(X=k) = 1 - [1.5854687x10^{-11} + 1.5864257x10^{-9}]$

$= 0.9999$

Therefore $\sum_{k=3}^{n=10} P(X=k) = 0.9999$

Exercise VI (2mks)

Let's find the algebraic form of the complex number $Z = \frac{Z_1}{Z_2} = \frac{(1+i)^4}{(\sqrt{3}-i)^3}$

Modulus of $Z_1 = \sqrt{1^2+1^2} = \sqrt{2}$ and argument $\theta = \tan^{-1}(1) = \frac{\pi}{4}$

Modulus of $Z_2 = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$ and argument $\varphi = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ Therefore

$\frac{Z_1}{Z_2} = \frac{\left[\sqrt{2}(cos\frac{\pi}{4}+sin\frac{\pi}{4})\right]^2}{\left[2\left(\cos\left(-\frac{\pi}{6}\right)+\sin(-\frac{\pi}{6})\right)\right]^3} = \frac{2\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)}{8\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)\hat{}-1} = \frac{1}{2}\left[\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)^1\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)^1\right]$

$\frac{Z_1}{Z_2} = \frac{1}{2}\left(cos\frac{\pi}{2}+sin\frac{\pi}{2}\right)^2 = \frac{1}{2}(cos\pi + sin\pi) = -\frac{1}{2}i$

Therefore we have $Z = -\frac{1}{2}i$