

SOLUTIONS
JULY 2009

Exercise I (4mks)

We are given two sequences (U_n) defined by $U_0 = 2, U_1 = 3$ and $U_n = \frac{4U_{n-1} - U_{n-2}}{3}$, and (V_n) defined by $V_n = U_n - U_{n-1}$, where $n \in \mathbb{N}^*$

a) Let's show that (V_n) is a geometrical sequence and determine the first term and the common ratio.

* We have $U_n = \frac{4U_{n-1} - U_{n-2}}{3} \dots \dots \dots (i), V_n = U_n - U_{n-1}, n \in \mathbb{N} \dots \dots \dots (ii).$

From equation (ii) we have $V_{n-1} = U_{n-1} - U_{n-2} \dots \dots \dots (iii)$

Substituting equation (i) into equation (ii) gives $V_n = \frac{4U_{n-1} - U_{n-2}}{3} - U_{n-1} = \frac{U_{n-1} - U_{n-2}}{3} = \frac{1}{3} V_{n-1} \dots (iv)$

From (iii). Therefore, $V_n = \frac{1}{3} V_{n-1}$

* For $n = 1$, and substituting in (ii) gives $V_1 = U_1 - U_0 = 3 - 2 = 1$

* From (iv) the common ratio is $\frac{1}{3}$

Therefore

From $V_n = \frac{1}{3} V_{n-1}$ shows that V_n is a gp. first term 1, common ratio $\frac{1}{3}$

b) Let's calculate the general term in terms of n .

Generally we have

$V_n = ar^n, a = V_1 = \text{first term}, r = \frac{1}{3} \text{ common ratio and substituting from (a) gives}$

$$V_n = ar^n = V_1 r^n = 1 \cdot \left(\frac{1}{3}\right)^n = 3^{-n}.$$

Therefore

General term $V_n = 3^{-n}, \forall n \in \mathbb{N}$

c) Let's calculate $S_n = V_1 + V_2 + V_3 + \dots + V_n$ in terms of n

$$\begin{aligned} \text{We have } S_n &= V_1 + V_2 + \dots + V_{n-1} + V_n = (U_1 - U_0) + (U_2 - U_1) + \dots + U_n - U_{n-1} \\ &= U_n - U_0 = \frac{a(1-r^n)}{1-r} = \frac{V_1(1-r^n)}{1-r} = \frac{1\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}} = \frac{1}{2}(3 - 3^{1-n}) \end{aligned}$$

$$S_n = \frac{1}{2}(3 - 3^{1-n}), \forall n \in \mathbb{N}$$

Therefore,

d) Let's show that the sequence (U_n) , converge and specify it's limit.

$$S_n = U_n - U_0 = \frac{1}{2}(3 - 3^{1-n}), \text{ hence } U_n = 2 + \frac{1}{2}(3 - 3^{1-n}) = \frac{1}{2}(7 - 3^{1-n})$$

$$U_{n+1} = \frac{1}{2}(7 + 3^{1-(1+n)}) = \frac{1}{2}(7 + 3^{-n}), U_{n+1} - U_n = 3^{-n} > 0 \dots (e), \therefore U_n \text{ is strictly increasing.}$$

$$* \lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} \left[\frac{1}{2}(7 + 3^{1-n}) \right] = \frac{7}{2} \dots \dots \dots (f)$$

since U_n is strictly increasing and $\lim_{n \rightarrow +\infty} U_n = \frac{7}{2}$, U_n converges to $\frac{7}{2}$

Therefore,

Exercise II (3mks)

1) Let's evaluate, $\int_0^{\cosh^{-1}(2)} \frac{\tanh x}{1 + \cosh x} dx$, given that $u = 1 + \cosh x$ leaving our answer in natural logarithm. $\tanh x = \frac{\sinh x}{\cosh x}$.

Let $U(x) = 1 + \cosh(x) = 1 + \frac{e^x + e^{-x}}{2}$, then $U(0) = 1 + \frac{e^0 + e^{-0}}{2} = 2$ and $U(\cosh^{-1}(x)) = 1 + \cosh(\cosh^{-1}(x)) = 3$, but $U - 1 = \cosh(x)$ and

$$\begin{aligned} dU &= \sinh(x)dx \\ \text{Thus } H &= \int_0^{\cosh^{-1}3} \frac{\tanh x}{1 + \cosh x} dx = \int_2^3 \frac{\sinh x}{\cosh x(1 + \cosh x)} dx \\ &= \int_2^3 \frac{dU}{(U-1)U} = \int_2^3 \left(\frac{1}{U-1} - \frac{1}{U} \right) dU \text{ by partial fractions} \\ H &= [\ln|U-1|]_2^3 - [\ln|U|]_2^3 \\ &= (\ln 2 - \ln 1) - (\ln 3 - \ln 2) = \ln \frac{4}{3} \end{aligned}$$

$$H = \ln \frac{4}{3}$$

2) Let's prove that $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ and hence or otherwise show that $\int_0^{\frac{1}{2}} \tanh^{-1} x dx = \frac{1}{4} \ln \left[\frac{27}{16} \right]$

* Let $y = \tanh^{-1}(x)$ then we have $x = \tanh y$ and $\text{sech}^2(y) \frac{dy}{dx} = 1$ but $\text{sech}^2(y) = 1 - \tanh^2(y)$ Therefore $\frac{dy}{dx} = \frac{d}{dx}(\tanh^{-1} x) = \frac{1}{\text{sech}^2(y)} = \frac{1}{1 - \tanh^2(y)} = \frac{1}{1 - x^2}$

* By using integration by parts, let's show that $\int_0^{\frac{1}{2}} \tanh^{-1}(x) dx = \frac{1}{4} \ln \left[\frac{27}{16} \right]$

$$\begin{aligned} \text{We let } \begin{cases} v(x) = \tanh^{-1}(x) \\ u'(x) = 1 \end{cases} &\Rightarrow \begin{cases} v'(x) = \frac{1}{(1-x^2)} \\ u(x) = x \end{cases} \\ \Rightarrow \int_0^{\frac{1}{2}} \tanh^{-1}(x) dx &= [x \tanh^{-1}(x)]_0^{\frac{1}{2}} - \frac{1}{2} \int_0^{\frac{1}{2}} \left[-\frac{2x}{(1-x^2)} \right] dx \\ &= \frac{1}{2} \tanh^{-1} \left(\frac{1}{2} \right) + \frac{1}{2} [\ln|1-x^2|]_0^{\frac{1}{2}} = \frac{1}{2} \tanh^{-1} \left(\frac{1}{2} \right) + \frac{1}{2} \ln \left(\frac{3}{4} \right) \\ &= \frac{1}{2} \cdot \frac{1}{2} \ln \left[\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right] + \frac{1}{2} \ln \left(\frac{3}{4} \right) = \frac{1}{4} \ln(3) + \frac{1}{2} \ln \left(\frac{3}{4} \right) \\ &= \frac{1}{4} \left[\ln 3 + 2 \ln \left(\frac{3}{4} \right) \right] = \frac{1}{4} \left[\ln 3 + \ln \left(\frac{9}{16} \right) \right] = \frac{1}{4} \ln \left[\frac{27}{16} \right] \end{aligned}$$

Therefore

$$\boxed{\frac{d(\tanh^{-1}(x))}{dx} = \frac{1}{(1-x^2)} \text{ and } \int_0^{\frac{1}{2}} \tanh^{-1}(x) dx = \frac{1}{4} \ln \left[\frac{27}{16} \right]}$$

Exercise III (4mks)

Let's determine the Cartesian equation of the plane, τ passing through;

a) $P(-2; 6; 7)$ and has a normal vector $\vec{n}(0; 3; 0)$:

Let $M(x; y; z)$ be a point on the plane, then $\overrightarrow{PM} = \begin{bmatrix} x+2 \\ y-6 \\ z-7 \end{bmatrix}$, $\vec{n} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, $\overrightarrow{PM}$ is perpendicular to τ hence.

$$\text{Implying that } \overrightarrow{PM} \cdot \vec{n} = 0 \Leftrightarrow \begin{bmatrix} x+2 \\ y-6 \\ z-7 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 3(y-6) = 0.$$

Thus $y = 6$

b) $P(-6; 10; 16)$ And its perpendicular to the right hand side of AB .

Given, $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}; B \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}; AB = \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix}; \text{Let } M \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \text{ then } \overrightarrow{PM} = \begin{bmatrix} x+6 \\ y-10 \\ z-16 \end{bmatrix}, \tau \text{ is perpendicular to } AB \text{ hence to } \overrightarrow{PM}. \text{ Thus } \overrightarrow{PM} \cdot \overrightarrow{AB} = 0 \Rightarrow \begin{bmatrix} x+6 \\ y-10 \\ z-16 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ -3 \end{bmatrix} = 0 \text{ giving us, } 2(x+6) - 3(y-10) - 3(z-16) = 0$

Thus Cartesian Equation of the plane is $2x - 3y - 3z + 90 = 0$

Exercise iv (4mks)

1) Let's find the square root $a + bi$ of the complex number $Z_1 = 5 + 12i, \sqrt{z}$

$$\begin{cases} a^2 + b^2 = |z_1| \\ a^2 - b^2 = \operatorname{Re}(z_1) \\ 2ab = \operatorname{Im}(z_1) \end{cases} \Leftrightarrow \begin{cases} a^2 + b^2 = 13 \dots \dots \dots (i) \\ a^2 - b^2 = 5 \dots \dots \dots (ii) \\ 2ab = 11 \dots \dots \dots (iii) \end{cases}$$

$$(i) + (ii) = 2a^2 = 18 \Rightarrow a = \pm 3$$

$$(i) - (ii) = 2b^2 = 8 \Leftrightarrow b = \pm 2 \quad \text{Thus}$$

$z_A = 3 + 2i \text{ and } z_B = -3 - 2i$

2) Let's find the modulus and argument of the complex number $z_2 = \frac{(1+i)^2}{(-1+i)^4}$

$$\begin{aligned} z_2 &= (1+i)^2(-1+i)^{-4} = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^2 \sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)^{-4} \\ &= \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^2 \left[\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right) \right]^{-4} \\ &= 2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{\frac{1}{4}} (\cos \pi + i \sin \pi) = \frac{1}{2} i (-1) \end{aligned}$$

Therefore

$z_2 = -\frac{1}{2}i$

3) Let's calculate $z_3 z_3^*$ and $\frac{z_3}{z_3^*}$ if $z_3 = 1 + i\sqrt{3}$

$$\text{*We have } z_3 z_3^* = (1 + i\sqrt{3})(1 - i\sqrt{3}) = 1 + 3 = 4$$

$$\text{*We have } \frac{z_3}{z_3^*} = \frac{(1+i\sqrt{3})}{(1-i\sqrt{3})} = \frac{(1+i\sqrt{3})(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})} = -\frac{1}{2} + i\frac{\sqrt{3}}{4},$$

$z_3 z_3^* = 4 \text{ and } \frac{z_3}{z_3^*} = -\frac{1}{2} + i\frac{\sqrt{3}}{4}$

Therefore,

Exercise v (3mks)

Let's calculate the following quantities

a) $\int \frac{3x+7}{x^2+x+1} dx$

$$\begin{aligned} \partial \text{ and } \varphi \quad \text{Therefore } \int \frac{3x+5}{x^2+x+1} dx &= \int \frac{\frac{3}{2}(2x+1) + \frac{7}{2}}{x^2+x+1} dx = \int \frac{\frac{3}{2}(2x+1)}{x^2+x+1} dx + \int \frac{\frac{7}{2}}{x^2+x+1} dx \\ &= \frac{3}{2} \ln(x^2 + x + 1) + \frac{7}{2} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C, C \in \mathbb{R}. \end{aligned}$$

Thus,

$\int \frac{3x+5}{x^2+x+1} dx = \frac{3}{2} \ln(x^2 + x + 1) + \frac{7}{2} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C, C \in \mathbb{R}$

b) $\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx$

$$\text{Let } 3x+7 = \gamma(-2x^2+x+1)' + \rho \quad (\gamma, \rho \in \mathbb{R})$$

$$= \gamma(-4x + 1) + \rho, \text{ therefore } \begin{cases} \rho = \frac{-3}{4} \\ \gamma = \frac{31}{4} \end{cases} \text{ by equating coefficients and solving for } \gamma \text{ \& } \rho.$$

$$\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx = \int \frac{\frac{-3}{4}(-4x+1) + \frac{31}{4}}{\sqrt{-2x^2+x+1}} dx = \frac{-3}{4} \int \frac{(-4x+1)}{\sqrt{-2x^2+x+1}} dx + \frac{31}{4} \int \frac{1}{\sqrt{-2x^2+x+1}} dx = -\frac{3}{4}A + \frac{31}{4}B$$

Let $U^2 = -2x^2 + x + 1$, then $2UdU = (-4x + 1)dx$ substituting in A gives us

$$\frac{-3}{4}A = -\frac{3}{4} \int \frac{2UdU}{\sqrt{U^2}} = -\frac{3}{4} \int 2dU = -\frac{3}{2}U + K' = -\frac{3}{2}\sqrt{-2x^2+x+1} + K'$$

$$\frac{31}{4}B = \frac{31}{4} \int \frac{1}{\sqrt{-2x^2+x+1}} dx = \frac{31}{4\sqrt{2}} \int \frac{1}{\sqrt{\frac{9}{16} - (x - \frac{1}{4})^2}} dx, \text{ by completing the square}$$

$$= \frac{31}{4\sqrt{2}} \int \frac{1}{\sqrt{\left[\frac{3}{4}\right]^2 + \left[x - \frac{1}{4}\right]^2}} dx = \frac{31}{4\sqrt{2}} \arcsin\left(\frac{4x-1}{3}\right) + K''$$

$$\boxed{\int \frac{3x+7}{\sqrt{-2x^2+x+1}} dx = \frac{-3}{4}\sqrt{-2x^2+x+1} + \frac{31}{4\sqrt{2}} \sin^{-1}\left(\frac{4x-1}{3}\right) + K, K = K' + K'' \in \mathbb{R}}$$

Exercise 6

Let's consider the probability density function "p. d. f." defined given by

$$f(x) = \begin{cases} cx^3(1-x^2), & \text{if } 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Let's calculate the value of the constant c

A continuous random variable X will be a probability density function iff

$$f(x) \geq 0, \text{ and } c \int_{-\infty}^{+\infty} f(x) dx = 1,$$

$$f(x) \geq 0 \Leftrightarrow cx^3(1-x^2) \geq 0 \Rightarrow c \geq 0, x^3 \geq 0 \text{ and } (1-x^2) \geq 0$$

$$\int_0^1 cx^3(1-x^2) dx = \int_0^1 cx^3 dx - \int_0^1 cx^5 dx = 1, \text{ since } x \in [0; 1]$$

$$= \frac{c}{4} [x^4]_0^1 - [x^6]_0^1 = 1 \Leftrightarrow \frac{c}{4} - \frac{c}{6} = \frac{c}{12} = 1 \text{ and } c = 12$$

$$\boxed{c = 12}$$

b) Let's find the mean and variance of X

*Mean, (X) of X,

$$\begin{aligned} \text{we have } E(X) &= \int_0^1 xf(x) dx = \int_0^1 x[12x^3(1-x^2)] dx = 12 \int_0^1 (x^4 - x^6) dx \\ &= \frac{12}{5} [x^5]_0^1 - \frac{12}{7} [x^7]_0^1 = \frac{12}{5} - \frac{12}{7} = \frac{24}{35} \end{aligned}$$

Therefore

$$\boxed{E(X) = \frac{24}{35}}$$

* The value of Var(X),

$$\text{we have } Var(X) = E(x^2) - [E(X)]^2$$

$$E(x^2) = \int_0^1 x^2 f(x) dx = \int_0^1 x^2 \cdot 12x^3 [1-x^2] dx = \int_0^1 12(x^5 - x^7) dx = 12 \left[\frac{x^6}{6} - \frac{x^8}{8} \right]_0^1 = \frac{12}{6} - \frac{12}{8} = \frac{1}{2}$$

$$\Leftrightarrow E(x^2) - [E(X)]^2 = \frac{1}{2} - \left(\frac{24}{35}\right)^2 = \frac{73}{2230},$$

Therefore,

$$\boxed{Var(X) = \frac{37}{2230}}$$

c) Let's determine the mode of X

$$\text{from p. d. f., We have } f(x) = [12x^3 - 12x^5] = 0 \Leftrightarrow f'(x) = 12x^2(3 - 5x^2) = 0$$

$$\therefore \text{ Either } 12x^2 = 0 \Rightarrow x = 0, \text{ or } 3 - 5x^2 = 0 \Leftrightarrow x = \pm \sqrt{\frac{3}{5}}$$

$$\text{but } x \in [0; 1] \Leftrightarrow x > 0 \Rightarrow x = \sqrt{\frac{3}{5}}$$

$$\boxed{\text{The mode } x = \sqrt{\frac{3}{5}}} \quad \therefore$$

d). Let's show that $6m^4 - 4m^6 + 1 = 0$, given m as a median of X

If X is a continuous random variable in the range $x_0 \leq x \leq x_k$ $k \in \mathbb{R}$, then we have the median, m as the value of X if $P(x_0 \leq x \leq x_k) = \frac{1}{2}$

$$\text{i. e. } \int_{x_0}^{x_k} f(x) dx = \frac{1}{2}$$

$$\Rightarrow \int_0^m (12x^3 - 12x^5) dx = \frac{1}{2}$$

$$\Rightarrow (3x^4 - 2x^6)_0^m = \frac{1}{2} \Rightarrow 6m^4 - 4m^6 = 1 \text{ and then we have}$$

$$\boxed{6x^4 - 4x^6 - 1 = 0} \quad \blacksquare$$