**Solution of Mathematics Entrance Examination**
**for ALL General Education 2012-2013**

**Exercise**

Let $\int_0^{\frac{\pi}{2}} \cos^n(t)dt$ for all whole numbers $\geq 1$.

1. Calculate $I_0, I_1$ and $I_2$.

$$\int_0^{\frac{\pi}{2}} \cos^0(t)dt = \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \cos^1(t)dt = [\sin(t)]_0^{\frac{\pi}{2}} = 1$$

$$\int_0^{\frac{\pi}{2}} \cos^2(t)dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos(2t)}{2}dt = [\frac{1}{2}t + \frac{1}{4}\sin(2t)]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Let's show that for all $n \geq 2$, we have: $nI_n = (n+1)I_{n-2}$.

$$\begin{cases} u'(t) = \sin(t)\cos^{n-2}(t) \\ v(t) = \sin(t) \end{cases} \Longrightarrow \begin{cases} u(t) = -\frac{1}{n-1}\cos^{n-1}(t) \\ v'(t) = \cos(t) \end{cases}$$

Hence, we have:

$$I_n = I_{n-2} - [\frac{1}{n-1}\sin(t)\cos^{n-1}(t)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{n-1}\cos^n(t)dt = I_{n-2} - \frac{1}{n-1}I_n$$

$$(n-1)I_{n-1} = (n-1)I_{n-2} - I_n$$

$$nI_n = (n-1)I_{n-2}$$

$$\boxed{\text{Thus for all whole number } r \geq 2\text{, we have } nI_n = (n-1)I_{n-2}}$$

2. Let's deduce the value of $I_n$ for all $n \geq 1$

For $n$ even we have $n = 2p$ ($p \in \mathbb{N}^*$), we have:

$$2pI_{2p} = (2p-1)I_{2p-2}$$

$$2I_2 = I_0$$

$$4I_4 = 3I_2$$

$$6I_6 = 5I_4$$

$$8I_8 = 7I_6$$

$$\vdots$$

$$2pI_{2p} = (2p-1)I_{2p-2}$$

After multiplication side by side we have:

$$\begin{aligned}
I_{2p} &= \frac{(2p-1)\times\ldots\times5\times3\times1}{2p\times\ldots\times6\times4\times2\times1}I_0\\
&= \frac{2p(2p-1)(2p-2)(2p-3)\times\ldots\times5\times3\times1}{(2p)^2(2p-1)(2p-2)^2\times\ldots\times6^2\times5\times4^2\times3\times2^2\times1}I_0\\
&= \frac{2p!(2p-1)(2p-3)\times\ldots\times5\times3\times1}{(2p)!(2p)(2p-2)\times\ldots\times4\times2\times1}I_0\\
&= \frac{(2p-1)(2p-3)\times\ldots\times5\times3\times1}{2^p p!}I_0\\
&= \frac{2p(2p-1)(2p-2)\times\ldots\times5\times4\times3\times2\times1}{2^p p!(2p)(2p-2)\times\ldots\times2\times1}I_0\\
&= \frac{2p!}{2^{2p}[p!]^2}I_0
\end{aligned}$$

$$\boxed{\text{Thus, } I_{2p} = \frac{2p!}{2^{2p}[p!]^2}I_0}$$

For $n$ odd we have $n = 2p+1$ ($p \in \mathbb{N}^*$) we have:

$$\begin{aligned}
(2p+1)I_{2p+1} &= (2p)I_{2p-1}\\
3I_3 &= 2I_1\\
5I_5 &= 4I_3\\
7I_7 &= 6I_5\\
&\vdots\\
(2p+1)I_{2p+1} &= 2pI_{2p-1}
\end{aligned}$$

After multiplication side by side we have:

$$\begin{aligned}
I_{2p+1} &= \frac{2p(2p-2)\times\ldots\times6\times4\times2\times1}{(2p+1)(2p-1)\times\ldots\times5\times3\times1}I_1\\
&= \frac{(2p)^2(2p-1)(2p-2)^2\times\ldots\times6^2\times5\times4^2\times3\times2^2\times1}{(2p+1)(2p)(2p-1)^2(2p-2)\times\ldots\times5^2\times4\times3^2\times2\times1}I_1\\
&= \frac{2p!(2p)(2p-2)\times\ldots\times6\times4\times2\times1}{(2p+1)!(2p-1)(2p-3)\times\ldots\times5\times3\times1}I_1\\
&= \frac{(2p)!2^p p!(2p)(2p-2)(2p-4)\times\ldots\times1}{(2p+1)!(2p-1)(2p)(2p-2)(2p-3)(2p-4)\times\ldots\times1}I_1\\
&= \frac{(2p)!2^{2p}(p!)^2}{(2p+1)!(2p)!}I_1\\
&= \frac{2^{2p}(p!)^2}{(2p+1)!}I_1
\end{aligned}$$

$$\boxed{\text{Thus, } I_{2p+1} = \frac{2^{2p}(p!)^2}{(2p+1)!}I_1}$$

**Exercise** 1. Let's study the continuity of the following function

$$f(x)=\begin{cases} x^2 & \text{if} \quad x\leq 0,\\ x & \text{if} \quad 0<x<2,\\ 4-x & \text{if} \quad x\geq 2.\end{cases}$$

We have: $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-} x^2=0$ and $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+} x^2=0$.

Thus $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)=0$, hence $f$ is continuous at 0.

Further, we have: $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 0^-} x=2$ and $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(4-x)=2$.

Thus $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)=f(2)=2$, hence $f$ is continuous at 2.

Therefore the function is continuous in this domain.

2. Let's show that function $g$ is continuous and differentiable and its derivative is continuous.

$$g(x)=\begin{cases} e^{\frac{1}{x}} & \text{if} \quad x<0,\\ 0 & \text{if} \quad x=0,\\ \cos(x)-1 & \text{if} \quad x>0.\end{cases}$$

- If $x<0$, $g'(x)=-\frac{1}{x^2}e^{\frac{1}{x}}$ let's pose $\frac{1}{x}=X$ when $x\longrightarrow 0^-$, $X\longrightarrow -\infty$.
  So, $\lim\limits_{x\to +\infty}(-X^2e^X)=0$.
- If $x>0$, $g'(x)=-\sin x$ and $g'(0)=0$.
- $x=0$, $g'(0)=0$.

Hence $g$ is differentiable in his domain and thus continuous in this same domain.



**Exercise** 1. A discret random variable $X$ has the following distribution of probability.

| $X=x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $P(X=x)$ | 0.1 | $p$ | 0.25 | $q$ | 0.05 |

$E(X)=2$

We find $p$ and $q$.

$$\begin{aligned} E(X) &= \sum x_i p_i \\ &= 0\times(0.1)+1p+2\times(0.25)+3q+4(0.05) \\ &= p+3q+0.7 \end{aligned} \tag{1}$$

$$\sum p_i = 1 \iff 0.1 + p + 0.25 + q + 0.05 = 1$$
$$\iff p + q = 0.6 \qquad (2)$$

$$\begin{cases} p + q = 0.6 \\ P + 3q = 1.3 \end{cases}$$

$$\boxed{q = 0.35 \text{ and } p = 0.25}$$

2. We find $V(X)$

$$V(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = 0^2 \times (0.1) + 1^2 \times (0.25) + 3^2 \times (0.35) + 4^2 \times (0.05) = 5.2$$

$$V(X) = 5.2 - 4$$

$$\boxed{\text{Thus, } V(X) = 1.2}$$

3. We find the average and variance of $Y = 5X + 7$

$$\begin{aligned} E(Y) &= E(5X+7) \\ &= 5E(X) + 7 \\ &= 5 \times 2 + 7 \\ &= 17 \end{aligned}$$

$$\boxed{\text{Thus, } E(Y) = 17}$$



$$\begin{aligned} V(Y) &= V(5X+7) \\ &= 5^2 V(X) \\ &= 25 \times 1.2 \\ &= 30 \end{aligned}$$

$$\boxed{\text{Thus, } V(Y) = 30}$$

**Exercise**

**Part I**

1. Let's show that exists a single triplet $(a, b, c)$, that one will determine, such that for any real $x$: $f(x) = ax + b + \frac{cx}{3x^2+1}$ for all $x$ belonging to $\mathbb{R}$. $f(x) = \frac{3(x-1)^3}{3x^2+1} = \frac{3x^3-9x^2+9x-3}{3x^2+1}$ after operating, this Euclidian division we obtain:

$$f(x) = x - 3 + \frac{8x}{3x^2+1}$$

   so by identification $a = 1, b = -3, c = 8$.

2. Limits

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(x - 3 + \frac{8x}{3x^2+1}) = +\infty$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x - 3 + \frac{8x}{3x^2+1}) = -\infty$$

3. Show that $f$ is differentiable and calculate its derivative.

   For all $x$ belonging to $\mathbb{R}$, $f$ is differentiable as sum of differentiable function in $\mathbb{R}$, i.e the polynomial function $x \longmapsto x - 3$ and the rational function $x \longmapsto \frac{8x}{3x^2+1}$.

   For all $x$ belonging to $\mathbb{R}$, $f'(x) = \frac{8(3x^2+1)-6x(8x)}{(3x^2+1)^2}$

   Thus, for all $x$ belonging to $\mathbb{R}$, $f'(x) = \frac{9x^4-18x^2+9}{(3x^2+1)^2}$.

   $f'(x) = 0$ is equivalent to $9x^4 - 18x^2 + 9 = 0$ is equivalent to $x^4 - 2x^2 + 1 = 0$.

   Let pose $X^2 = x^4$. $f'(x) = 0$ is equivalent to $X^2 - 2X + 1 = 0$.

   We obtain $x_1 = x_2 = 1$ and $f(1) = 0$.

   So for all $x$ belonging to $\mathbb{R}$, $f$ is strictly increasing.

4. Let draw the table of variation of $f$.

| $x$ | $-\infty$ | $1$ | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | | $+$ | |
| $f(x)$ | $-\infty$ | $0$ | $+\infty$ |

5. Let's show that the curve $(C)$ has an asymptote oblique the line $(D)$ of equation $y = x-3$.

   For all $x$ belonging to $\mathbb{R}$, we have:

$$\lim_{x\to\pm\infty}[f(x)-(x-3)] = \lim_{x\to\pm\infty}\left(\frac{8x}{3x^2+1}\right) = 0.$$

   Hence, $(D): y = x-3$ is an asymptote to the curve $(C)$.

6. Lets study the relative positions of $(C)$ and $(D)$.

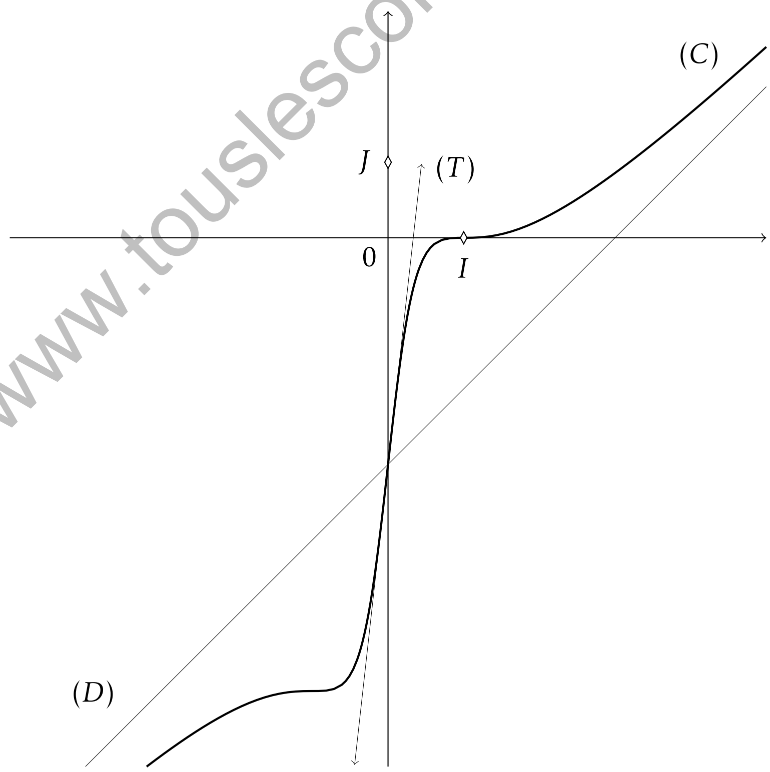
   For all $x$ belonging to $\mathbb{R}$, we have $u(x) = f(x)-(x-3) = \frac{8x}{3x^2+1}$

   $u(x) = 0$ is equivalent to $8x = 0$ i.e $x = 0$

   Hence, for all $x \leq 0$, $(C)$ is bellow the line $(D)$ and for all $x \geq 0$, $(C)$ is above the line $(D)$.

7. Let's give the equation of the tangent $(T)$ to $(C)$ at the point of $x$–coordinate 0. Draw $(T),(C)$ and $(D)$.

   By definition, $(T): y = f'(x_0)(x-x_0)+f(x_0)$ with $x_0 = 0$. Hence, $(T): y = 9x-3$ is the line tangent to the curve $(C)$.




8. Let's show that the curve $(C)$ has a center of symmetry.

   The curve $(C)$ intersect the oblique asymptote $(D)$ at the point $\omega(0;-3)$ and more over, the function $s(x) = f(x-0)-3 = f(x)-3$ is odd $(s(-x) = -s(x))$.

Therefore, $\omega(0;-3)$ is the center of symmetry of the curve $(C)$.

9. Let's that the equation $f(x) = 1$ has a single solution in $\mathbb{R}$. One notes $\alpha$ solution.

   Lets define $v$ the function such that, $v(x) = f(x) - 1$, so $v'(x) = f'(x)$.

   Thus, $v$ is continuous and strictly increasing in $\mathbb{R}$, so realize a bijection from $\mathbb{R} \to \mathbb{R}$, moreover, $v(-\infty).v(+\infty) < 0$, therfore there exist a single solution $\alpha$ belonging to $\mathbb{R}$, such that $v(x) = 0$ i.e such that $f(x) = 1$.

10. Let's give the approximate value of $\alpha$ to $10^{-2}$ near by excess.

    By projection on the curve, we have $\alpha$ belongs to $]3;4[$ and $f(3) = 0.857$ and $f(4) = 1.653$ and 1 belongs to $[0.857;1.653]$ by bisection method, we obtain $f(3.20) = 1$, Hence $\alpha = 3.20$.

**Part II**

1. Let's show that $g$, is differentiable on $\mathbb{R}$ and lets calculate $g'(x)$.

   $g$ is differentiable on $\mathbb{R}$ as the composition of differentiable functions on $\mathbb{R}$ i.e $t(x) = \sin(x)$ and $f(x) = \frac{3(x-1)^2}{3x^2+1}$.
   so, $g(x) = (tof)(x)$; thus $g'(x) = f'(x) \cdot t'of(x)$ with $t'(x) = \cos(x)$ and $f'(x) = \frac{9x^4-18x^2+9}{(3x^2+1)^2}$.

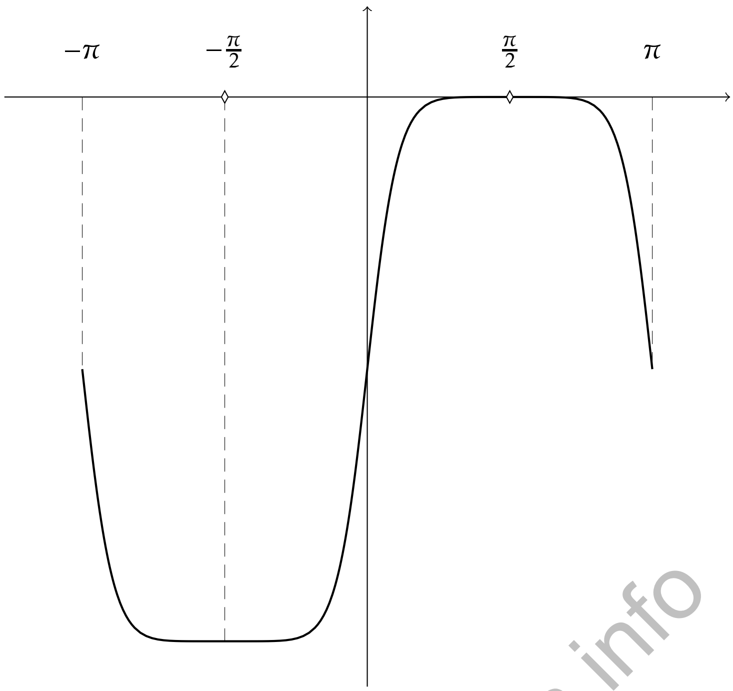
   Hence,
   $$g'(x) = \cos(x)\left(\frac{9\sin^4(x) - 18\sin^2(x) + 9}{(3\sin^2(x) + 1)^2}\right)$$

2. Lets draw the table of variation of $g$ on $[-\pi;\pi]$.

   $g'(x) = 0$ is equivalent to $x = \pm\frac{\pi}{2}$ with $g(\frac{\pi}{2}) = 0$ and $g(-\frac{\pi}{2}) = -6$.

| $x$ | $-\pi$ | | $-\frac{\pi}{2}$ | | $\frac{\pi}{}2$ | | $\pi$ |
|---|---|---|---|---|---|---|---|
| $g'(x)$ | | $-$ | 0 | $+$ | 0 | $-$ | |
| $g(x)$ | $-3$ | $\searrow$ | $-6$ | $\nearrow$ | 0 | $\searrow$ | $-3$ |

3. Lets plot on a new drawing the curve representative of $g$.


$-\pi$
$-\frac{\pi}{2}$
$\frac{\pi}{2}$
$\pi$