Mathematics Entrance Examination for BAC C, D, E and A/L

General Education

Electrical and power engineering, civil engineering, mechanial engineering

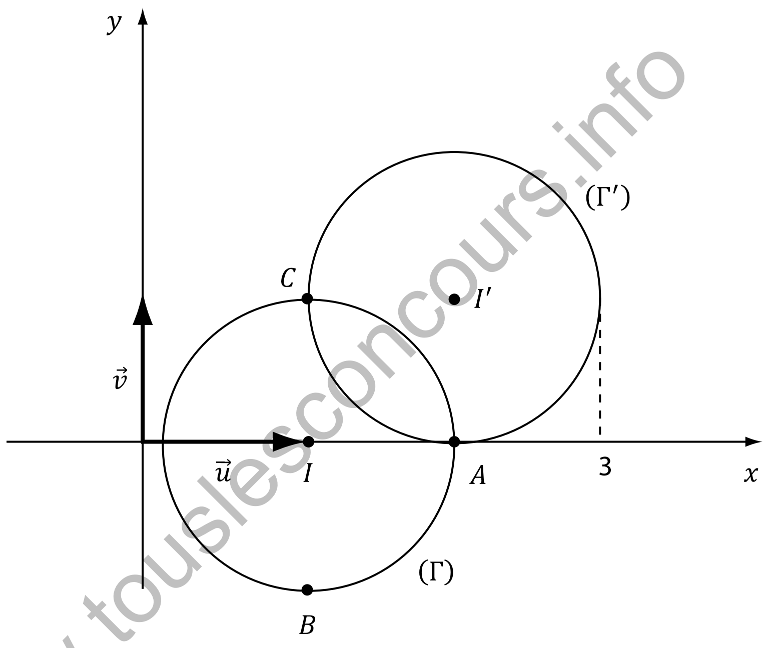
Entrance Examination

Exercise

In the complex plane with an orthogonal line $(O, \vec{u}, \vec{v})$ (graphic unit: 2cm).

We have: $A(2)$, $B(1-i)$ and $C(i+1)$ with $a = 2, b = 1-i$ and $c = 1+i$.

1. (a) A, B, C on a figure.



(b) Calculation of $\frac{c-a}{b-a}$.

$$\frac{c-a}{b-a} = \frac{(1+i)-2}{1-i-2} = \frac{-1+i}{-1-i} = \frac{(-1+i)(-1-i)}{2} = \frac{1-i-i-1}{2} = -i$$

Thus

$$\frac{c-a}{b-a} = -i = e^{-i\frac{\pi}{2}}$$

Furthermore, we have $| \frac{c-a}{b-a} =|= |-i| = 1$

This implies that $\frac{AC}{AB} = 1 \Longleftrightarrow AC = AB$

We have $AB = AC$ and $\arg(\widehat{\overrightarrow{AC}, \overrightarrow{AB}}) = \frac{\pi}{2}$.

Thus ABC is a right angle isosceles triangle at A.

2. (a) Determination of θ the angle of the rotation r of center A. We call r = rotation (A, θ), from the hypothesis $r(B) = C$.

let calculate d affix of D such that $D = r(C)$.

$$\begin{cases} r(A) = A \\ r(B) = C \end{cases} \iff \begin{cases} AB = AC \\ mes(\widehat{\overrightarrow{AB}, \overrightarrow{AC}}) = \theta \end{cases} \iff \begin{cases} z_A = az_A + b \quad (1) \\ z_{C'} = az_B + b \quad (2) \end{cases}$$

$$\begin{aligned} (1)-(2) &\iff 2-(1+i) = 2a - a(1-i) \\ &\iff 2-1-i = 2a - a + ai \\ &\iff 1-i = a(1+i) \\ &\iff a = \frac{(1-i)(1-i)}{2} = -i \end{aligned}$$

$$(1) \Longrightarrow 2 = 2(-i) + b \Longrightarrow b = 2+2i \quad \text{thus,} \quad z' = -iz + 2 + 2i$$

$$z_D = -iz_C + 2 + 2i = -i(1+i) + 2 + 2i = -i + 1 + 2 + 2i = 3 + i$$

thus,

$$\boxed{\text{d=3+i}}$$

(b) Let (Γ) the circle with diameter $[BC]$. We determine and construct $(\Gamma') = r(\Gamma)$.
$\begin{cases} r(B) = C \\ r(C) = D \end{cases}$ let I' the center of the diameter $[CD]$.

$$z_{I'} = \frac{c+d}{2} = \frac{1+i+3+i}{2} = 2+i$$

thus $z_{I'} = 2 + i$.

The radius of (Γ') is $R' = \frac{1}{2}|d-c| = \frac{1}{2}|3+i-1-i| = \frac{1}{2}|2| = 1$.

Hence, $(\Gamma') = \mathcal{C}(I', R' = 1)$.

3. Let $M \in (\Gamma)$ $\quad (M \neq C)$ and $M' \in (\Gamma')$.

$$r : M \longrightarrow M'$$

$$z \longmapsto z' = r(z)$$

(a) Let prove that there exist $\theta \in [0; \frac{\pi}{2}[\cup]\frac{\pi}{2}; 2\pi]$ such that $z = 1 + e^{i\theta}$.

Let J the center of diameter $[BC]$.

$$z_I = \frac{b+c}{2} = \frac{1-i+1+i}{2} = 2$$

thus $(\Gamma) = \xi(I, R)$. thus,

$$R = \frac{1}{2}BC = \frac{1}{2}|c-b| = \frac{1}{2}|1+i-1+i| = \frac{1}{2}|2i|$$

$$R = 1$$

$$(\Gamma) = \xi(I, R = 1).$$

$$M \in (\Gamma) \text{ iff } \begin{cases} IM = R \\ mes(\widehat{\overrightarrow{IB}, \overrightarrow{IM}}) = \theta \end{cases}$$

$$\frac{z - z_I}{z_B - z_I} = e^{i\theta} \Longrightarrow z - z_I = (z_B - z_I)e^{i\theta}.$$

$$z = (z_B - z_I)e^{i\theta} + z_I \Longrightarrow z_B = 1 + i, z_J = 1$$

$$z - z_I = e^{i\theta}, z = e^{i\theta} + z_I$$

Thus,

$$z = 1 + e^{i\theta}$$

(b) Expression of z in function of θ.

$$z = 1 + e^{i\theta} = e^{i\frac{\theta}{2}} \times e^{-i\frac{\theta}{2}} + e^{i\theta} = (e^{-i\frac{\theta}{2}} + e^{i\frac{\theta}{2}})(e^{i\frac{\theta}{2}})$$

$$z = 2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}$$

$$\theta \in [0; \frac{\pi}{2}[\cup]\frac{\pi}{2}; 2\pi]$$

(c) Let prove that $\frac{z'-c}{z-c}$ is real: we have $\frac{z'-c}{z-c} = \frac{-iz+1+i}{z-(1+i)}$ since $z' = -iz + 2 + 2i = \frac{-i(z-(1-i))}{z-(1+i)}$

$$\begin{aligned}
\arg(\frac{z'-c}{z-c}) &= \arg[\frac{-i(z-(1-i))}{z-(1+i)}][2\pi] \\
&= [\arg(-i) + \arg[\frac{(z-(1-i))}{z-(1+i)}]][2\pi] \\
&= -\frac{\pi}{2} + \arg(z-(1-i)) - \arg(z-(1+i)) \\
&= -\frac{\pi}{2} + (\widehat{\overrightarrow{MC}, \overrightarrow{MB}})(2\pi) \\
&= -\frac{\pi}{2} + \frac{\pi}{2}(2\pi) \text{ since } (\widehat{\overrightarrow{MC}, \overrightarrow{MB}}) = \frac{\pi}{2}
\end{aligned}$$

Thus $\arg(\frac{z'-c}{z-c}) = 0(2\pi)$ therefore $\frac{z'-c}{z-c}$ is real.

Method 2

$$\begin{aligned}
\text{since } z' &= \frac{-iz+2+2i-1-i}{z-1-i} \\
&= \frac{-iz+1+i}{z-(1+i)} \\
&= \frac{-i-ie^{i\theta}+1+i}{z-1-i} \\
&= \frac{-iz+1+i}{z-(1+i)} \\
&= \frac{-i-ie^{i\theta}+1+i}{1+e^{i\theta}-1-i} \\
&= \frac{1-ie^{i\theta}}{-i+e^{i\theta}} \\
&= \frac{1+\sin\theta-i\cos\theta}{\cos\theta+i(1+i\sin\theta)} \\
&= \frac{[(1+\sin\theta)-i\theta][\cos\theta-i(-1+\sin\theta)]}{\cos^2\theta+(-1+\sin\theta)^2} \\
&= \frac{(1+\sin\theta)\cos\theta-i(1-\sin\theta)-i\cos^2\theta-\cos\theta(-1+\sin\theta)}{\cos^2\theta+(-1+\sin\theta)^2} \\
&= \frac{\cos\theta[1+\sin\theta+1-\sin\theta]-i[-1+\sin\theta-\sin^2\theta+\cos^2\theta]}{\cos^2\theta+(-1+\sin\theta)^2} \\
&= \frac{2\cos\theta}{\cos^2\theta+(-1+\sin\theta)^2} \\
&= \frac{2\cos\theta}{1-2\sin\theta} \text{ which is real}
\end{aligned}$$

So we deduce that, there exist $\lambda \in \mathbb{R}$ such that $\frac{z'-c}{z-c} = \lambda \Longleftrightarrow \frac{CM'}{CM} = \lambda \Longleftrightarrow CM' = \lambda CM$

thus M, M' and C are on the same line.

Exercise

Suppose $y'' + 5y' = 0 \ (E)$.

1. Let's show that f is a solution of (E) if and only if $F = f'$ is solution of $5y' + 5y = 0 \ (E_1)$.

   Let's suppose f' is a solution of $y' + 5y = 0 \ (E_1)$.

   Moreover f' being solution of $(E_1) \Longleftrightarrow F' + 5F = 0 \Longleftrightarrow f'' + 5f' = 0 \Longleftrightarrow f$ is the solution of (E).

   Thus f' is a solution of $(E_1) \Longleftrightarrow f$ is a solution of (E).

2. Computing (E).

   $(E) : f'' + 5f' = 0$

Let the auxiliary equation be $r^2 + 5r + 0 = 0$.

$\Delta = 5^2 - 4(1)(0) = 25$

$r = \frac{-5\pm\sqrt{\Delta}}{2} \Longrightarrow \begin{cases} r_1 = \frac{-5+5}{2} \\ r_2 = \frac{-5-5}{2} \end{cases} \Longrightarrow \begin{cases} r_1 = 0 \\ r_2 = -5 \end{cases}$ Hence: $f(x) = A + Be^{-5x}, A, B \in \mathbb{R}$

$$\boxed{f(x) = A + Be^{-5x}}$$

3. Let $g(x) = a\cos(x) + b\sin(x)$ be the complementary equation.

   $\forall x \in \mathbb{R}, g(x) = a\cos(x) + b\sin(x)$ where $a, b \in \mathbb{R}$.

   Computing a and b

   $\begin{cases} g(x) = a\cos(x) + b\sin(x) \\ g'(x) = (a\sin(x) + b\cos(x) \end{cases}$

$$\begin{aligned} (E') &\iff -a\cos(x) - b\sin(x) - 5a\sin(x) + 5\cos(x) = 26\cos(x) \\ &\iff \begin{cases} 5b - a = 26 \\ -b - 5a = 0 \end{cases} \\ &\iff \begin{matrix} a = -1 \\ b = 5 \end{matrix} \end{aligned}$$

   Thus, $\boxed{g(x) = -\cos(x) + 5\sin(x)}$ for every real number x.

4. Let's show that f is a solution of (E') if and only if $f - g$ is a solution of (E).

   Suppose $f - g$ is a solution of (E).

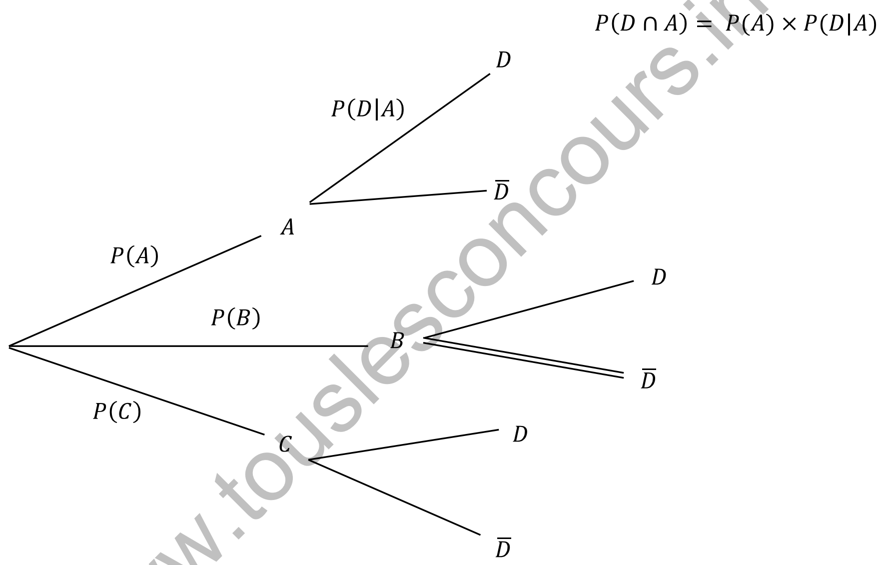
$$\begin{aligned} f - g \quad \text{is a solution of} \quad (E) &= (f-g)'' + 5(f-g)' = 0 \\ &= f'' + 5f' = g'' + 5g' \\ &= f'' + 5f' = 26`\cos(x) \\ &= f \quad \text{is a solution of} \quad (E) \end{aligned}$$

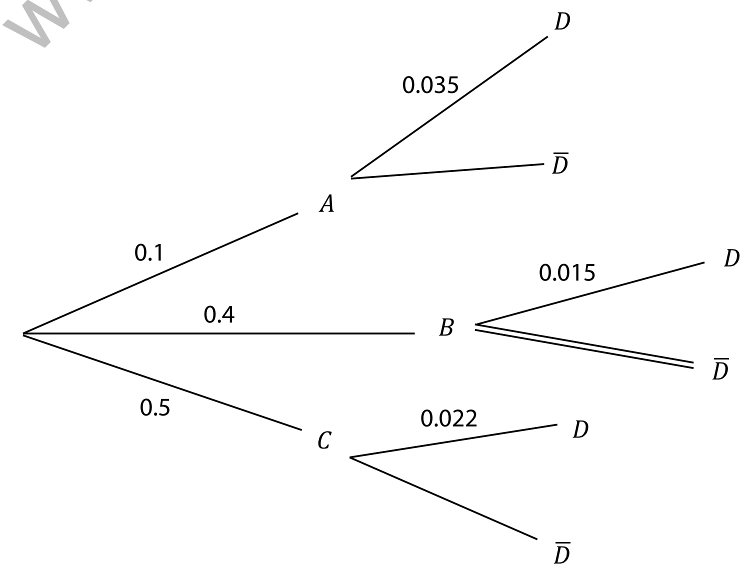
   Then, $(f - g)$ is a solution of $(E) \Longleftrightarrow f$ is a solution of (E').

5. Computation of the complete solution of (E'). From above, the particular solution $(P.S)$ is $f(x) = A + Be^{-5x} - \cos(x) + 5\sin(x), \forall x \in \mathbb{R}$.

6. Computing the particular solution verifying the conditions $f(0) = 0$ and $f'(0) = 0$.
$$\begin{cases} f(x) = A + Be^{-5x} - \cos(x) + 5\sin(x) \\ f'(x) = -5Be^{-5x} + \sin(x) + 5\cos(x) \end{cases}$$
$$f(0) = 0 \Longleftrightarrow \begin{cases} 1 + A = 1 \\ A = 0 \end{cases}$$
$$f'(0) = 0 \Longleftrightarrow \begin{cases} -5B + 5 = 0 \\ B = 1 \end{cases}$$
Thus, $\boxed{f(x) = e^{-5x} - \cos(x) + 5\sin(x), \forall x \in \mathbb{R}}$

Exercise





Suppose,

The event A: “Be the event the sweet chosen is from machine A”

The event B: “Be the event, the sweet chosen is from machine B”

The event C: “Be the event, the sweet chosen is from machine C”

The sweet D: “Be the event, the sweet is bad”

The sweet $\overline{D}$: “Be the event, the sweet is not bad”

1. Let’s compute the probability that sweet chosen is from machine C and it is bad is 0.011.

   But $\boxed{P(C \cap D) = P(C) \cap P(D|C)}$

   Thus, $P(C \cap D) = 0.5 \times 0.022 = 0.011$

   $\boxed{P(C \cap D) = 0.011}$

2. Let’s compute the probability that the sweet chosen is bad

   $\boxed{P(D) = P(A \cap D) + P(B \cap D) = P(C \cap D)}$

   As such $P(D) = 0.1 \times 0.035 + 0.4 \times 0.0015 + 0.5 \times 0.022$

   $\boxed{P(D) = 0.0205}$

3. Let’s compute probability that the sweet is from machine C given that it is bad:

   Since $P(C \cap D) = P(D|C) \times P(D) \Longrightarrow \boxed{P(D|C) = \frac{P(C \cap D)}{P(D)}}$

   $P(D|C) = \frac{0.011}{0.0205} = 0.53$

   $\boxed{P(D|C) = 0.53}$

4. Let us compute the probability of obtaining at least one bad (or defective) sweet among the 10 chosen sweet:

   Let ""be the numbers of bad sweet obtained when 4– probability of obtaining at least one defective sweet from tensamplings, pick successively from the tray

   $P(X = 0) + P(X = 1) + P(X = 2) + ... + P(X = 10) = 1$

   $P(X \geq 1) + P(X = 0) = 1$

   $P(X \geq 1) = 1 - P(X = 0)$

   But $P(X = D) = C_{10}^{10-k} D^{10-k} (\overline{D})^k = C_{10}^{10-k} D^{10-k} (1 - D)^k$

   $P(X = 0) = C_{10}^{0} (0.025)^{10-10} (1 - 0.025)^{10} = (1 - 0.025)^{10}$

Thus $P(X \geq 1) = 1 - P(X = 0) = (1 - (1 - 0.025)^{10}) = 0.22$

$\boxed{P(X \geq 1) = 0.22}$