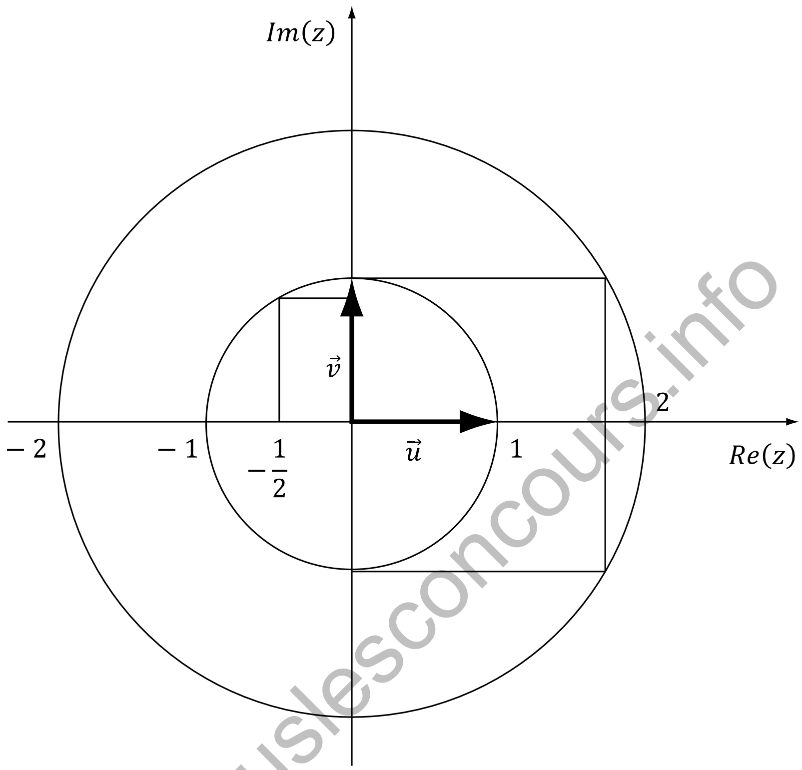
Civil and Forestry Engineering (technical section)

July 2009-2010

Exercise



$A\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$; $B\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix}$ and $D\begin{pmatrix}-\frac{1}{2}\\\frac{\sqrt{3}}{2}\end{pmatrix}$. Suppose $r(O,\frac{\pi}{3})$ is a rotation through an angle of $\frac{\pi}{3}$ about the center 0.

$t(1)$ is translation by a vector of affix 1.

Hence $\begin{cases} r(O,\frac{\pi}{3})(A) = A' \ \ (1) \\ r(O,\frac{\pi}{3})(B) = B' \ \ (2) \end{cases}$

$$\begin{aligned}(1) \Longleftrightarrow z_{A'} &= e^{i\frac{\pi}{3}} z_A \\ &= (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})(\sqrt{3}+i) \\ &= (\frac{1}{2} + i\frac{\sqrt{3}}{2})(\sqrt{3}+i) \\ &= 2i\end{aligned}$$

Thus, $\boxed{z_{A'} = 2i}$

Likewise, (2) $\Longleftrightarrow z_{B'} = e^{i\frac{\pi}{3}} z_B = \sqrt{3} + i$

$$\text{Thus, } \boxed{z_{B'} = \sqrt{3} + i}$$

The Images $z_{D'}$ of z_D by $t(1)$

$z_{D'} = z_D + 1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} + 1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$\text{Hence, } \boxed{z_{D'} = \frac{1}{2} + i\frac{\sqrt{3}}{2}}$$

It is quite evident plotting the points $z_{A'}$, $z_{B'}$, $z_{D'}$ on the faller plane

Computation of $\arg(\frac{z_{A'} - z_{B'}}{z_{D'}})$

$$\begin{aligned} z_s &= \frac{(2i) - (\sqrt{3} + i)}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} \\ &= 2\frac{-\sqrt{3} + i}{1 + i\sqrt{3}} \\ &= \frac{2(-\sqrt{3} + i)(1 - i\sqrt{3})}{1^2 + (\sqrt{3})^2} \\ &= 2i \end{aligned}$$

$$\boxed{z_s = 2i}$$

$$z_s = \begin{cases} |z_s| = 2 \\ \arg(z_s) = \frac{\pi}{2} \end{cases}$$

As a result $\arg(z_s) = \arg(\frac{z_{A'} - z_{B'}}{z_{D'}}) = \frac{\pi}{2}$.

Hence (OD') is medium of triangle $OA'B'$.

Exercise

Suppose $y'' + 5y' = 0 \ (E)$.

1. Let's show that f is a solution of (E) if and only if $F = f'$ is solution of $5y' + 5y = 0 \ (E_1)$.

   Let's suppose f' is a solution of $y' + 5y = 0 \ (E_1)$.

   Moreover f' being solution of $(E_1) \Longleftrightarrow F' + 5F = 0 \Longleftrightarrow f'' + 5f' = 0 \Longleftrightarrow f$ is the solution of (E).

   Thus f' is a solution of $(E_1) \Longleftrightarrow f$ is a solution of (E).

2. Computing (E).

   $(E) : f'' + 5f' = 0$

   Let the auxiliary equation be $r^2 + 5r + 0 = 0$.

$\Delta = 5^2 - 4(1)(0) = 25$

$r = \frac{-5\pm\sqrt{\Delta}}{2} \Longrightarrow \begin{cases} r_1 = \frac{-5+5}{2} \\ r_2 = \frac{-5-5}{2} \end{cases} \Longrightarrow \begin{cases} r_1 = 0 \\ r_2 = -5 \end{cases}$ Hence: $f(x) = A + Be^{-5x}, A, B \in \mathbb{R}$

$$\boxed{f(x) = A + Be^{-5x}}$$

3. Let $g(x) = a\cos(x) + b\sin(x)$ be the complementary equation.

   $\forall x \in \mathbb{R}, g(x) = a\cos(x) + b\sin(x)$ where $a, b \in \mathbb{R}$.

   Computing a and b

   $\begin{cases} g(x) = a\cos(x) + b\sin(x) \\ g'(x) = (a\sin(x) + b\cos(x) \end{cases}$

$$\begin{aligned} (E') &\iff -a\cos(x) - b\sin(x) - 5a\sin(x) + 5\cos(x) = 26\cos(x) \\ &\iff \begin{cases} 5b - a = 26 \\ -b - 5a = 0 \end{cases} \\ &\iff \begin{matrix} a = -1 \\ b = 5 \end{matrix} \end{aligned}$$

Thus, $\boxed{g(x) = -\cos(x) + 5\sin(x)}$ for every real number x.

4. Let's show that f is a solution of (E') if and only if $f - g$ is a solution of (E).

   Suppose $f - g$ is a solution of (E).

$$\begin{aligned} f - g \quad \text{is a solution of} \quad (E) &\iff (f-g)'' + 5(f-g)' = 0 \\ &\iff f'' + 5f' = g'' + 5g' \\ &\iff f'' + 5f' = 26`\cos(x) \\ &\iff f \quad \text{is a solution of} \quad (E) \end{aligned}$$

   Then, $(f - g)$ is a solution of $(E) \iff f$ is a solution of (E').

5. Computation of the complete solution of (E'). From above, the particular solution $(P.S)$ is $f(x) = A + Be^{-5x} - \cos(x) + 5\sin(x), \forall x \in \mathbb{R}$.

6. Computing the particular solution verifying the conditions $f(0) = 0$ and $f'(0) = 0$.
$$\begin{cases} f(x) = A + Be^{-5x} - \cos(x) + 5\sin(x) \\ f'(x) = -5Be^{-5x} + \sin(x) + 5\cos(x) \end{cases}$$
$$f(0) = 0 \Longleftrightarrow \begin{cases} 1 + A = 1 \\ A = 0 \end{cases}$$
$$f'(0) = 0 \Longleftrightarrow \begin{cases} -5B + 5 = 0 \\ B = 1 \end{cases}$$
Thus, $\boxed{f(x) = e^{-5x} - \cos(x) + 5\sin(x), \forall x \in \mathbb{R}}$

Exercise 1. True. $f(x) = ln(\frac{2x+1}{x-1})$. f exist if and only if $\begin{cases} x \neq -\frac{1}{2} \\ x \neq 1 \\ \frac{2x+1}{x-1} > 0 \end{cases} \Longleftrightarrow \begin{cases} 2x+1 \neq 0 \\ x-1 \neq 0 \\ \frac{2x+1}{x-1} > 0 \end{cases}$

x	$-\infty$		$-\frac{1}{2}$		1		$+\infty$
$2x+1$		$-$	0	$+$		$+$	
$x-1$		$-$		$-$	0	$+$	
$\frac{2x+1}{x-1}$		$+$	0	$-$	$\|\|$	$+$	

Thus, $D_f =]-\infty; -\frac{1}{2}[\cup]1; +\infty[$.

2. False. $ln'(\frac{2x+1}{x-1}) = \frac{-3}{(2x+1)(x-1)}$.

   $\forall x \in D_f, f'(x) = \frac{-3}{(2x+1)(x-1)}$.

3. True.
$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} [ln(\frac{2x+1}{x-1})] \\ &= +\infty \end{aligned}$$
   Hence, $x = 1$ is an asymptote the curve (C_f).

4. True. $\lim_{x\to\pm\infty} f(x) = 2$.

   This implies $y = 2$ is the horizontal asymptote to (C_f).

Solutions of Mathematics Entrance examination for BAC F, MA, CH, MAV, MEM
Electrical and power engineering, civil engineering, Mechanical engineering
July 2010-2011

Exercise

$A(3,0,1); B(0,-1,2)$; and $C(1,-1,0)$ are taken into consideration into $(O,\vec{i},\vec{j},\vec{k})$.

1. Coordinates of vector $\vec{n}$.

   ◇ $\vec{n} = \overrightarrow{AB} \wedge \overrightarrow{AC}$, but we have: $\overrightarrow{AB}\begin{pmatrix}-3\\-1\\-3\end{pmatrix}$ and $\overrightarrow{AC}\begin{pmatrix}-2\\-1\\-1\end{pmatrix}$

   So, $\overrightarrow{AB} \wedge \overrightarrow{AC} = (-2+0)\vec{i} - (3-6)\vec{j} + (0+1)\vec{k} = -2\vec{i} + 3\vec{j} + \vec{k}$.

   Thus, $\vec{n}(-2,3,1)$.

   ◇ Deduction of plane (ABC) cartesian equation.

   $\vec{n}$ is the normal vector of the plane (ABC) so for all $M\begin{pmatrix}x\\y\\z\end{pmatrix} \in (ABC)$, we have: $\overrightarrow{AM} \perp \vec{n}$.

$$\begin{aligned}\overrightarrow{AM} \perp \vec{n} &\iff \overrightarrow{AB} \times \vec{n} = 0\\ &\iff \begin{pmatrix}x-3\\y-0\\z-1\end{pmatrix}.\begin{pmatrix}-2\\3\\1\end{pmatrix} = 0\\ &\iff -2(x-3)+3(y)+(z-1)=0\\ &\iff -2x+3y+z+6-1=0\\ &\iff -2x+3y+z+5=0\end{aligned}$$

   Thus, the plane (ABC) equation is $(ABC): -2x+3y+z+5=0$ (1).

2. Let calculate the product: $\overrightarrow{DA}.(\overrightarrow{BD} \wedge \overrightarrow{DC})$ with $D(1,1,-2)$.

   We have: $\overrightarrow{DA}\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\overrightarrow{DB}(-1,-2,0)$, $\overrightarrow{DC}(0,-2,2)$. So, $\overrightarrow{BD} \wedge \overrightarrow{DC} = -4\vec{i} + 2\vec{j} + 2\vec{k}$.

Thus,

$$\begin{aligned}\overrightarrow{DA}\cdot(\overrightarrow{BD}\wedge\overrightarrow{DC}) &= (2\vec{i}-\vec{j}+3\vec{k})\cdot(-4\vec{i}+2\vec{j}+2\vec{k})\\ &= -8-2+6\\ &= -4\end{aligned}$$

Thus, $\overrightarrow{DA}\cdot(\overrightarrow{BD}\wedge\overrightarrow{DC}) = -4$.

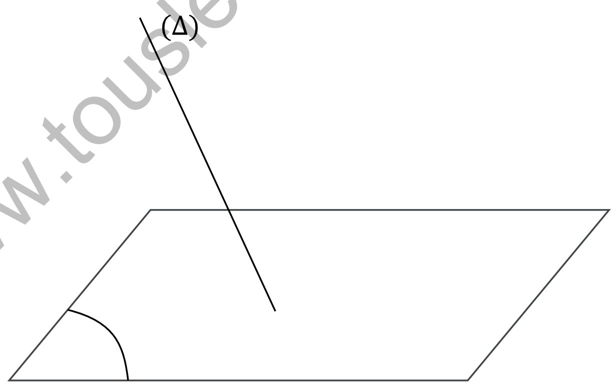
3. Parametric representative of the line through D with directory vector $\vec{n}$.

For every point $M(x,y,z)\in(\Delta)$, we have $\overrightarrow{DM}$ collinear to $\vec{n}$.

$\overrightarrow{DM}$ collinear to $\vec{n} \Longleftrightarrow \overrightarrow{DM} = \lambda\vec{n}$

$$\begin{aligned}\overrightarrow{DM} = \lambda\vec{n} &\Longleftrightarrow \begin{pmatrix} x-1\\ y-1\\ z+2\end{pmatrix} = \lambda\begin{pmatrix}-2\\ 3\\ 1\end{pmatrix}\\ &\Longleftrightarrow \begin{cases} x = -2\lambda+1\\ y = 3\lambda+1\\ z = \lambda-2\end{cases} \qquad (2)\end{aligned}$$

Coordinates of $H = (\Delta)\cap(ABC)$



Let fist determine λ. (2) into (1),

$$\begin{aligned}-2(-2\lambda+1)+3(3\lambda+1)+(\lambda-2)+5=0 &\Longleftrightarrow 4\lambda+9\lambda+\lambda-2+3+5-2=0\\ &\Longleftrightarrow 14\lambda=-4\\ &\Longleftrightarrow \lambda=-\frac{2}{7}\end{aligned}$$

Thus, $H = (\Delta)\cap(ABC) \Longleftrightarrow \begin{cases} x = -2(-\frac{2}{7})+1\\ y = 3(-\frac{2}{7})+1\\ z = -\frac{2}{7}-2\end{cases} \Longleftrightarrow \begin{cases} x = \frac{11}{7}\\ y = \frac{1}{7}\\ z = -\frac{16}{7}\end{cases}$

Calculation of $d(D,(ABC))$.

$$d(D,(ABC)) = \frac{|-2(1)+3(1)-2+5|}{\sqrt{(-2)^2+3^2+1^2}} = \frac{|4|}{\sqrt{14}} = \frac{2\sqrt{14}}{7}.$$

Thus, $d(D,(ABC)) = \frac{2\sqrt{14}}{7}$.

D' is the symmetrical image of D in relation to (ABC).

$$so\, \overrightarrow{DD'} = \frac{1}{2}\overrightarrow{DH} \iff \begin{cases} x_{D'} - x_D = \frac{1}{2}(x_H - x_D) \\ y_{D'} - y_D = \frac{1}{2}(y_H - y_D) \\ z_{D'} - z_D = \frac{1}{2}(z_H - z_D) \end{cases}$$

$$\iff \begin{cases} x_{D'} = \frac{1}{2}(x_H - x_D) \\ y_{D'} = \frac{1}{2}(y_H - y_D) \\ z_{D'} = \frac{1}{2}(z_H - z_D) \end{cases}$$

$$\iff \begin{cases} x_{D'} = \frac{1}{2}(\frac{11}{7} + 1) \\ y_{D'} = \frac{1}{2}(\frac{1}{7} + 1) \\ z_{D'} = \frac{1}{2}(-\frac{16}{7} + 2) \end{cases}$$

$$\iff \begin{cases} x_{D'} = \frac{9}{7} \\ y_{D'} = \frac{4}{7} \\ z_{D'} = -\frac{15}{7} \end{cases}$$

Thus, $D'(\frac{9}{7}; \frac{4}{7}; -\frac{15}{7})$

Exercise

See Civil, engineering 2009-2010 (technical section)

Exercise

$$f(x) = \frac{2e^x + 3}{e^x - 1} \iff D_f = \mathbb{R}^*$$

1.

$$\lim_{x \to +\infty} \frac{2e^x + 3}{e^x - 1} = \frac{e^x(2 + \frac{3}{e^x})}{e^x(1 + \frac{1}{e^x})}.$$

But,

$$\lim_{x \to +\infty} \frac{1}{e^x} = 0 \Longrightarrow \lim_{x \to +\infty} \frac{2e^x + 3}{e^x - 1} = \lim_{x \to +\infty} \frac{e^x(2 + \frac{3}{e^x})}{e^x(1 + \frac{1}{e^x})} = \lim_{x \to +\infty} \frac{2 + \frac{3}{e^x}}{1 + \frac{1}{e^x}} = 2.$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} \frac{2e^x+3}{e^x-1} = \frac{2e^{-\infty}+3}{e^{-\infty}-1} = \frac{3}{-1} = -3.$$

2.

$$\lim_{x\to 0^+} \left(\frac{2e^x+3}{e^x-1}\right) = \frac{5}{0^+} = +\infty$$
$$\lim_{x\to 0^-} \left(\frac{2e^x+3}{e^x-1}\right) = \frac{5}{0^-} = -\infty$$

3. To deduce the equation of asymptote.

   Since $\lim_{x\to 0} f(x) = \pm\infty$ the line $x = 0$ is a vertical asymptote.

   Also, $\lim_{x\to-\infty} f(x) = -3$ and $\lim_{x\to+\infty} f(x) = 2$.

   Therefore the lines $y = 2$ and $y = -3$ are (horizontal asymptote).

1. Let's determine the derivative of f. $f(x) = \frac{2e^x+3}{e^x-1}$

$$f'(x) = \frac{(2e^x-1)(2e^x)-(2e^x+3)(e^x)}{(e^x-1)^2} = \frac{4e^x-2e^x-2e^x-3e^x}{(e^x-1)^2} = \frac{-5e^x}{(e^x-1)^2}$$

   f is strictly decreasing.

x	$-\infty$	0	$+\infty$
$f'(x)$	$-$	‖	$-$
$f(x)$	$-3 \searrow -\infty$	‖	$+\infty \searrow 2$

2. Let's prove that $\Omega(0;-\frac{1}{2})$ is the center of symmetry.

   Let $x = X + 0, y = Y - \frac{1}{2}$.

   if, $F(-X) = F(X)$, Ω is center of symmetry.

$f(x) = y = \frac{2e^x+3}{e^x-1}$, replacing coefficients

$$f(x) = y = \frac{2e^x+3}{e^x-1} - \frac{1}{2} = \frac{2e^x+3}{e^x-1} + \frac{1}{2}$$
$$Y = \frac{4e^x+6+e^x-1}{2(e^x-1)} = \frac{5e^x+5}{2(e^x-1)}$$
$$F(-x) = \frac{5}{2}\left(\frac{e^{-x}+1}{e^{-x}-1}\right) = \frac{5}{2}\left[\frac{\frac{1}{e^x}+1}{\frac{1}{e^x}-1}\right] = \frac{5}{2}\left[\frac{1+e^x}{1-e^x}\right] = -\frac{5}{2}\left[\frac{e^x+1}{e^x-1}\right]$$
$$F(x) = -\frac{5}{2}\left[\frac{e^x+1}{e^x-1}\right] = -F(x).$$

Therefore the point $\Omega(0;-\frac{1}{2})$ is the center of symmetry to the curve of f.